arXiv:1504.00299v1 [math.LO] 1 Apr 2015

ON EQUIVALENCE RELATIONS GENERATED BY SCHAUDER BASES

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Abstract. In this paper, a notion of Schauder equivalence relation $\mathbb{R}^{\mathbb{N}}/L$ is introduced, where L is a linear subspace of $\mathbb{R}^{\mathbb{N}}$ and the unit vectors form a Schauder basis of L. The main theorem is to show that the following conditions are equivalent:

(1) the unit vector basis is boundedly complete;
(2) L is F_σ in $\mathbb{R}^{\mathbb{N}}$;
(3) $\mathbb{R}^{\mathbb{N}}/L$ is Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$.

We show that any Schauder equivalence relation generalized by basis of ℓ_2 is Borel bireducible to $\mathbb{R}^{\mathbb{N}}/\ell_2$ itself, but it is not true for bases of c_0 or ℓ_1. Furthermore, among all Schauder equivalence relations generated by sequences in c_0, we find the minimum and the maximum elements with respect to Borel reducibility.

We also show that $\mathbb{R}^{\mathbb{N}}/\ell_p$ is Borel reducible to $\mathbb{R}^{\mathbb{N}}/J$ iff $p \leq 2$, where J is James' space.

1. Introduction

The notion of Borel reducibility becomes a tool to compare objects or problems from different branches of mathematics. In recent years, many equivalence relations concerning Banach space theory were investigated. One motivation of this paper is the Borel reducibility among equivalence relations $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0$. It is proved by Dougherty and Hjorth: for $p, q \geq 1$,

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q \iff p \leq q,$$

while $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are Borel incomparable (see [4] and [11]). These results were generalized via different methods by several authors (see, e.g., [18] and [3]). In this paper, we study equivalence relations of the form $\mathbb{R}^{\mathbb{N}}/L$, where L is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. Moreover, the class of of subspaces L we focus on in this paper can be specified by the following two equivalent conditions:

Date: April 2, 2015.

2010 *Mathematics Subject Classification.* Primary 03E15, 46B15, 46B45.

Research partially supported by the National Natural Science Foundation of China (Grant No. 11371203).

(i) there is a sequence (x_n) of none-zero elements in a Banach space X such that
$$L = \mathrm{coef}(X,(x_n)) \stackrel{\mathrm{Def}}{=} \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n)x_n \text{ converges in } X\}.$$
(ii) the unit vectors $e_n = (0,0,\cdots,0,\overset{n}{1},0,\cdots)$ form a Schauder basis of L.

If one, and thus all of above conditions hold for L, we call $\mathbb{R}^{\mathbb{N}}/L$ a Schauder equivalence relation.

Schauder equivalence relations were already studied in different disguises. For example, it is obvious that all $\mathbb{R}^{\mathbb{N}}/\ell_p$ ($p \geq 1$) and $\mathbb{R}^{\mathbb{N}}/c_0$ are Schauder equivalence relations. Most recently, equivalence relations $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}/\ell_p(\ell_q)$ were considered by Gao and Yin [8]. We can easily see that, for any $p, q \geq 1$, $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}/\ell_p(\ell_q)$ is Borel bireducible to a Schauder equivalence relation. With a continuous function $f : [0,1] \to \mathbb{R}^+$, Mátrai [18] defined a relation $\mathbf{E}_f$ on $[0,1]^{\mathbb{N}}$. Borel reducibility between equivalence relations of the form $\mathbf{E}_f$ were investigated in [18]. Yin [22] generalized Mátrai's results to show that the partial order structure $P(\omega)/\mathrm{Fin}$ can be embedded into the set of these $\mathbf{E}_f$'s equipped with the partial ordering of Borel reducibility. In fact, as noted in Yin [22], any such $\mathbf{E}_f$ appeared in [22] is Borel bireducible to a Schauder equivalence relation $\mathbb{R}^{\mathbb{N}}/L$, where L is an Orlicz sequence space.

The main theorem of this paper is the following:

Theorem 1.1. *If the unit vectors (e_n) form a Schauder basis of a Banach space L. Then the following are equivalent:*

(1) *(e_n) is boundedly complete basis;*
(2) *L is F_σ in $\mathbb{R}^{\mathbb{N}}$;*
(3) *$\mathbb{R}^{\mathbb{N}}/L \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$.*

In is well known that $\mathbb{R}^{\mathbb{N}}/\ell_p \sim_B [0,1]^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0 \sim_B [0,1]^{\mathbb{N}}/c_0$. We generalize these results to the following:

Theorem 1.2. *Let (x_n) be a symmetric basis of Banach space X. Then*
$$E(X,(x_n)) \sim_B [0,1]^{\mathbb{N}}/\mathrm{coef}(X,(x_n)).$$

Reducibility and nonreducibility between Schauder equivalence relations generated by different sequences of same space are considered, especially in case that the generating sequences are Schauder bases. In this paper, we mainly focus on bases of three special Banach spaces: ℓ_2, c_0, and ℓ_1.

Theorem 1.3. *For any basis (y_k) of ℓ_2, we have $E(\ell_2,(y_k)) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_2$.*

In contract, for c_0, we construct special bases (x_n^m) for each $m \geq 1$ and $m = \infty$, and denote $\mathrm{cs}^{(m)} = \mathrm{coef}(c_0,(x_n^m))$. For $m = 1$, we have
$$\mathrm{cs}^{(1)} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n) \text{ converges}\}.$$

We show that

Theorem 1.4. (1) *For $m \geq 1$, we have*
$$\mathbb{R}^{\mathbb{N}}/c_0 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m+1)} <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$
(2) *Let (x_n) be a none-zero sequence in c_0. Then*
$$\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$

While for ℓ_1, we construct a basis (y_n^1) with
$$\mathrm{coef}(\ell_1,(y_n^1)) = \mathrm{bv}_0 \stackrel{\mathrm{Def}}{=} c_0 \cap \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |a(n) - a(n+1)| < +\infty\},$$
and prove $\mathbb{R}^{\mathbb{N}}/\ell_1 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 <_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$, where $\otimes$ is the direct product operator between equivalence relations.

We also compare $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/J$ where J is James' space and get
$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/J \iff p \leq 2.$$

The paper is organized as follows. In section 2 we recall some notions in descriptive set theory and functional analysis, and introduce two kind of equivalence relaitons. In section 3 we prove Theorem 1.1. In section 4 we prove an useful lemma for converting a Borel reduction to an additive reduction. In section 5 we focus on Schauder equivalence relations generated by bases of ℓ_2, c_0, and ℓ_1. In section 6 we prove Theorem 1.2 and compare $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/J$. Finally section 7 contains some further remarks and open problems.

2. Preliminaries

A *Polish space* is a separable completely metrizable topological space. Let E and F be equivalence relations on Polish spaces X and Y respectively. A Borel function $\theta : X \to Y$ is called a *Borel reduction* from E to F if, for any $x, y \in X$,
$$(x,y) \in E \iff (\theta(x), \theta(y)) \in F.$$
We say E is *Borel reducible* to F, denoted $E \leq_B F$, if there exists a Borel reduction from E to F. If both $E \leq_B F$, $F \leq_B E$ hold, we say E and F are *Borel bireducible*, denoted $E \sim_B F$. We also denote $E \leq_B F$ and $F \not\leq_B E$ as $E <_B F$. We refer to [7] and [14] for background of Borel reducibility.

A sequence (x_n) in a Banach space X is called a *Schauder basis* (or basis, for the sake of brevity) of X if, for any $x \in X$, there is a unique sequence $a \in \mathbb{R}^{\mathbb{N}}$ such that $x = \sum_n a(n)x_n$. Let (x_n) be a Schauder basis of X. Define $P_n : X \to X$ as $P_n(\sum_n a(n)x_n) = \sum_{i \leq n} a(n)x_n$. Then all P_n are bounded and the *basis constant* $\sup_n \|P_n\| < +\infty$. It follows that, there is a sequence (x_n^*) of bounded linear functional on X, such that $x = \sum_n x_n^*(x)x_n$. We call (x_n^*) the *biorthogonal functionals* associated to (x_n).

Let (x_n) be a basis of X. We say (x_n) is *unconditional* if, for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, the sequence $(x_{\pi(n)})$ is also a basis. A basis (x_n) is said to be *boundedly complete* if, for every sequence $a \in \mathbb{R}^{\mathbb{N}}$ such that $\sup_n \|\sum_{i \leq n} a(i)x_i\| < \infty$, the series $\sum_n a(n)x_n$ converges.

Let (x_n) be a sequence of none-zero elements in a Banach space X. The closed linear span of $\{x_n : n \in \mathbb{N}\}$ is denoted by $[x_n]_{n\in\mathbb{N}}$. We denote

$$\mathrm{coef}(X,(x_n)) = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n)x_n \text{ converges}\},$$

and for $a \in \mathrm{coef}(X,(x_n))$, we define

$$\|a\| = \sup_n \|\sum_{i\leq n} a(i)x_i\|.$$

By Cauchy criterion, it is routine to check that $(\mathrm{coef}(X,(x_n)), \|\cdot\|)$ is a Banach space, and the unit vectors $e_n = (0,0,\cdots,0,\overset{n}{1},0,\cdots)$ form a basis of it. From the definition of $\mathrm{coef}(X,(x_n))$, we can easily see that, if X is a closed subspace of Y, then $\mathrm{coef}(X,(x_n)) = \mathrm{coef}(Y,(x_n))$.

A sequence (x_n) is called *normalized* if $\|x_n\| = 1$ for each n, and is called *semi-normalized* if there are $A \geq B > 0$ such that $A \geq \|x_n\| \geq B$ for each n. It is easy too see that, for a semi-normalized sequence (x_n) of X, we always have $\ell_1 \subseteq \mathrm{coef}(X,(x_n)) \subseteq c_0$.

We say two sequences (x_n) and (y_n) of X are *equivalent* if $\mathrm{coef}(X,(x_n)) = \mathrm{coef}(X,(y_n))$. A basis (x_n) of X is said to be *symmetric* if, for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $(x_{\pi(n)})$ is equivalent to (x_n). All symmetric basis are actually unconditional.

Definition 2.1. Let (x_n) be a sequence in a Banach space X. We define an equivalence relation on $\mathbb{R}^{\mathbb{N}}$ as $E(X,(x_n)) = \mathbb{R}^{\mathbb{N}}/\mathrm{coef}(X,(x_n))$, i.e., for all $a, b \in \mathbb{R}^{\mathbb{N}}$,

$$(a,b) \in E(X,(x_n)) \iff a - b \in \mathrm{coef}(X,(x_n)).$$

We call this kind of equivalence relations *Schauder equivalence relations*.

A none-zero sequence (x_n) of a Banach space X is said to be a *basic sequence* if it is a basis of $[x_n]_{n\in\mathbb{N}}$.

Let (x_n) be a basis of X, (r_n) a sequence of real numbers, and $0 = n_0 < n_1 < \cdots$ an strictly increasing natural numbers. If for every k, $u_k = \sum_{n=n_k}^{n_{k+1}-1} r_n x_n$ is not 0, we call sequence (u_k) a *block basis* of (x_n). A block basis is no necessarily a basis, but is always a basic sequence. A simple reduction θ witnesses that $E(X,(u_k)) \leq_B E(X,(x_n))$ defined as, for any $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, $\theta(a)(n) = a(k)r_n$ for $n_k \leq n < n_{k+1}$. For any sequence (r_n) of non-zero numbers, we always have $E(X,(x_n)) \sim_B E(X,(r_n x_n))$. Therefore, we may assume any basis is normalized when we need.

Let (x_n) be a semi-normalized basis of X, (r_n) a sequence of positive real numbers with $\sum_n r_n < +\infty$. Since $\ell_1 \subseteq \mathrm{coef}(X,(x_n))$, we have

$$E(X,(x_n)) \sim (\prod_n r_n\mathbb{Z})/\mathrm{coef}(X,(x_n)).$$

A desired Borel reduction θ defined as $\theta(a)(n) = r_n[a(n)/r_n]$ for $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Let E and F be two equivalence relations on X and Y respectively. Denote $E \otimes F$ the equivalence relation on $X \times Y$ as

$$((x_1, y_1), (x_2, y_2)) \in E \otimes F \iff (x_1, x_2) \in E \ \&\ (y_1, y_2) \in F$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Let (x_n) be a sequence in X and (y_n) a sequence in Y. We denote $z_{2k} = x_k$ and $z_{2k+1} = y_k$ for $k \in \mathbb{N}$. It is easy to see that

$$E(X \oplus Y, (z_n)) \sim_B E(X, (x_n)) \otimes E(Y, (y_n)).$$

So we may think $E(X, (x_n)) \otimes E(Y, (y_n))$ is still a Schauder equivalence relation. Furthermore, if (x_n) and (y_n) are Schauder bases on X and Y respectively, we can see that (z_n) is also a basis on $X \oplus Y$.

A sequence (X_n) of closed subspaces of a Banach space X is called a *Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum_n x_n$, with $x_n \in X_n$ for each n. Similar to Schauder bases, define $P_n : X \to X$ as $P_n(\sum_n x_n) = \sum_{i \le n} x_i$ where $x_n \in X_n$ for each n. Then all P_n are bounded and the *decomposition constant* $\sup_n \|P_n\| < +\infty$.

Definition 2.2. Let (X_n) be a Schauder decomposition of a separable Banach space X. We define an equivalence relation $E(X, (X_n))$ on $\prod_n X_n$ as

$$(\alpha, \beta) \in E(X, (X_n)) \iff \sum_n (\alpha(n) - \beta(n)) \text{ converges in } X$$

for any $\alpha, \beta \in \prod_n X_n$. We call this kind of equivalence relations *decomposition equivalence relations.*

Furthermore, if all these X_n are finite dimensional, we call $E(X, (X_n))$ an *F.D.D. equivalence relation.*

For more details for Schauder bases and Schauder decompositions, we refer to [16]. A tiny difference on notation with [16] is, in this paper, any sequence (x_n) means $(x_0, x_1, \cdots)$, not $(x_1, x_2, \cdots)$.

3. F_σ Schauder equivalence relations

In the light of Rosendal's Theorem that any K_σ equivalence relation on a Polish space is Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$ (see [21]), we compare F_σ Schauder equivalence relations and $\mathbb{R}^{\mathbb{N}}/\ell_\infty$.

The following lemma will be used to convert a Borel reduction to a continuous reduction.

Lemma 3.1. *Denote $D = \{d \in \mathbb{R}^{\mathbb{N}} : \forall n (4^n d(n) \in \mathbb{Z})\}$. Let G be a dense G_δ set in D, $a \in \mathbb{R}^{\mathbb{N}}$ with $2^n a(n) \in \mathbb{Z}$ for each $n \in \mathbb{N}$, and let $-1 = n_0 < n_1 < \cdots < n_k < \cdots$. Then there exist $b^* \in D$ and a strictly increasing sequence of natural numbers (k_l) with $k_0 = 0$ such that $G \supseteq C$, where*

$$C = \{d \in D : \forall l \exists i \le 2^l \forall n \in (n_{k_l}, n_{k_{l+1}}] (d(n) = b^*(n) + i a(n)/2^l)\}.$$

Proof. Assume that $O_0 \supseteq O_1 \supseteq \cdots \supseteq O_l \supseteq \cdots$ be a sequence of dense open sets with $G = \bigcap_l O_l$. We will construct b^* and (k_m) by induction on m such that $O_m \supseteq C_m$, where

$$C_m = \{d \in D : \forall l \leq m \exists i \leq 2^l \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + ia(n)/2^l)\}.$$

When we finish the construction, we shall have

$$C = \bigcap_m C_m \subseteq \bigcap_m O_m = G.$$

First, for $m = 0$, fix a $d_0^0 \in O_0$. Since O_0 is open, there is n_0^0 such that $O_0 \supseteq \{d \in D : \forall n \leq n_0^0(d(n) = d_0^0(n))\}$. Set $b^*(n) = d_0^0(n)$ for $n \leq n_0^0$. Denote

$$N_1^0 = \{d \in D : \forall n \leq n_0^0(d(n) = d_0^0(n) + a(n))\}.$$

Since O_0 is dense, there is a $d_1^0 \in N_1^0 \cap O_0$. Then we can find an $n_1^0 \geq n_0^0$ such that $O_0 \supseteq \{d \in D : \forall n \leq n_1^0(d(n) = d_1^0(n))\}$. Select k_1 such that $n_{k_1} \geq n_1^0$ and set $b^*(n) = d_1^0(n) - a(n)$ for $n_0^0 < n \leq n_{k_1}$. Now we denote

$$C_0 = \{d \in D : \exists i \in \{0, 1\} \forall n \leq n_{k_1}(d(n) = b^*(n) + ia(n))\}.$$

Then $C_0 \subseteq O_0$.

Secondly, assume that we have defined $k_1, \cdots, k_m$ and the value of $b^*(n)$ for $n \leq n_{k_m}$. Let $s_0, s_1, \cdots, s_J$ be an enumeration of the following set

$$\{(i_0, \cdots, i_m) : \forall l \leq m(0 \leq i_l \leq 2^l)\}.$$

We inductively find a sequence $n_0^m < n_1^m < \cdots < n_J^m$ as follows.

Denote

$$N_0^m = \{d \in D : \forall l < m \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + s_0(l)a(n)/2^l)\}.$$

Since O_m is dense, there is $d_0^m \in N_0^m \cap O_m$. Then since O_m is open, we can find an $n_0^m \geq n_{k_m}$ such that $O_m \supseteq \{d \in D : \forall n \leq n_0^m(d(n) = d_0^m(n))$. Set $b^*(n) = d_0^m(n) - s_0(m)a(n)/2^m$ for $n_{k_m} < n \leq n_0^m$.

Further assume that n_j^m and the value of $b^*(n)$ for $n \leq n_j^m$ have been defined.

If $j < J$, denote by N_{j+1}^m the set of all $d \in D$ satisfying

$$d(n) = \begin{cases} b^*(n) + s_{j+1}(l)a(n)/2^l, & n \in (n_{k_l}, n_{k_{l+1}}] \text{ for } l < m, \\ b^*(n) + s_{j+1}(m)a(n)/2^m, & n_{k_m} < n \leq n_j^m \end{cases}$$

By the same reason, we can find $d_{j+1}^m \in N_{j+1}^m \cap O_m$ and $n_{j+1}^m \geq n_j^m$ with $O_m \supseteq \{d \in D : \forall n \leq n_j^m(d(n) = d_{j+1}^m(n))\}$. Then set $b^*(n) = d_{j+1}^m - s_{j+1}(m)a(n)/2^m$ for $n_j^m < n \leq n_{j+1}^m$.

If $j = J$, select a k_{m+1} such that $n_{k_{m+1}} \geq n_J^m$. Set $b^*(n) = d_J^m(n) - s_J(m)a(n)/2^m$ for $n_J^m < n \leq n_{k_{m+1}}$. It easy to see $O_m \supseteq C_m$ as desired. $\square$

The next theorem is slightly more general then Theorem 1.1.

Recall that a Schauder decomposition (X_n) of a Banach space X is called *boundedly complete* if, for every sequence $\alpha \in \prod_n X_n$ such that $\sup_n \|\sum_{i \leq n} \alpha(i)\| < +\infty$, the series $\sum_n \alpha(n)$ converges (see [16]).

Theorem 3.2. *Let (X_n) be a Schauder decomposition of a separable Banach space X. Then the following are equivalent:*

(1) *(X_n) is boundedly complete;*
(2) *$\mathrm{cs}(X,(X_n)) \stackrel{\mathrm{Def}}{=} \{\alpha \in \prod_n X_n : \sum_n \alpha(n)$ converges$\}$ is F_σ in $\prod_n X_n$;*
(3) *$E(X,(X_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$.*

Proof. Define $S : X \to \prod_n X_n$ as $S(x) = (x_n)$ for $x = \sum_n x_n$ with each $x_n \in X_n$. Because all projections P_n of the decomposition (X_n) are bounded, we see S is a continuous injection whose range is $\mathrm{cs}(X,(X_n))$. Let M be the decomposition constant $\sup_n \|P_n\|$. For $m \geq 1$, we denote

$$\begin{aligned} B_m &= \{\alpha \in \textstyle\prod_i X_i : \sup_n \|\sum_{i\leq n} \alpha(i)\| \leq m\} \\ &= \textstyle\bigcap_n \{\alpha \in \prod_i X_i : \|\sum_{i\leq n} \alpha(i)\| \leq m\}. \end{aligned}$$

Then B_m is closed in $\prod_n X_n$.

(1)⇒(2). From the definition of boundedly complete decomposition, we have

$$\mathrm{cs}(X,(X_n)) = \bigcup_m B_m,$$

so $\mathrm{cs}(X,(X_n))$ is F_σ.

(2)⇒(1). Assume that $\mathrm{cs}(X,(X_n)) = \bigcup_m F_m$ with each F_m closed in $\prod_n X_n$. Then each $S^{-1}(F_m)$ is closed in X. By Baire category theorem, there exits an m such that $S^{-1}(F_m)$ has an inner point. So there exist $y^\# \in X$ and $r > 0$ such that

$$S^{-1}(F_m) \supseteq B(y^\#, r) = \{x \in X : \|x - y^\#\| \leq r\}.$$

Now for any sequence $\alpha \in \prod_n X_n$ with $\sup_n \|\sum_{i\leq n} \alpha(i)\| < +\infty$, without loss of generality, we may assume that $\sup_n \|\sum_{i\leq n} \alpha(i)\| \leq r$. For each $j \in \mathbb{N}$, we define $\alpha_j \in \prod_n X_n$ as

$$\alpha_j(n) = \begin{cases} \alpha(n), & n \leq j, \\ 0, & n > j. \end{cases}$$

Then $\alpha_j \in \mathrm{cs}(X,(X_n))$ and $\|S^{-1}(\alpha_j)\| \leq r$. Therefore, for $j \in \mathbb{N}$, we have $y^\# + S^{-1}(\alpha_j) \in B(b_0, r) \subseteq S^{-1}(F_m)$, i.e., $S(y^\#) + \alpha_j \in F_m$. Note that, in $\prod_n X_n$, $\lim_{j\to\infty} \alpha_j \to \alpha$. Since F_m is closed in $\prod_n X_n$, we have $S(y^\#) + \alpha \in F_m \subseteq \mathrm{cs}(X,(X_n))$. Hence $\alpha \in \mathrm{cs}(X,(X_n))$.

(1)⇒(3). Let $\{U_k : k \in \mathbb{N}\}$ be a basis for the topology of $\prod_n X_n$. For $\alpha \in \prod_n X_n$ and $k \in \mathbb{N}$, since $\bigcup_m (\alpha + B_m) = \alpha + \mathrm{cs}(X,(X_n))$ is dense in $\prod_n X_n$, there are some m such that $(\alpha + B_m) \cap U_k \neq \emptyset$. So we can define

$$\theta(\alpha)(k) = \min\{m : (\alpha + B_m) \cap U_k \neq \emptyset\}.$$

It is easy to see that $\theta : \prod_n X_n \to \mathbb{R}^{\mathbb{N}}$ is Borel.

For $\alpha, \beta \in \prod_n X_n$, if $(\alpha,\beta) \in E(X,(X_n))$, let $\|S^{-1}(\alpha - \beta)\| \leq K/M$ with $K \in \mathbb{N}$. Then $\alpha - \beta \in B_K$, so for each m, $\alpha + B_m \subseteq \beta + B_{m+K}$, and $\beta + B_m \subseteq \alpha + B_{m+K}$. It follows that $|\theta(\alpha)(k) - \theta(\beta)(k)| \leq K$ for each k,

and hence $\theta(\alpha) - \theta(\beta) \in \ell_\infty$. On the other hand, if $(\alpha, \beta) \notin E(X, (X_n))$, then

$$(\alpha + \mathrm{cs}(X, (X_n))) \cap (\beta + \mathrm{cs}(X, (X_n))) = \emptyset.$$

Thus for each $l \geq 1$, we have $(\alpha + B_1) \cap \bigcup_{m \leq l} (\beta + B_m) = \emptyset$. We can find a k such that $\alpha + B_1$ meets U_k, and $(\beta + B_m) \cap U_k = \emptyset$ for all $m \leq l$, so $|\theta(\alpha)(k) - \theta(\beta)(k)| \geq l$. It follows that $\theta(\alpha) - \theta(\beta) \notin \ell_\infty$.

Therefore, θ is a Borel reduction of $E(X, (X_n))$ to $\mathbb{R}^\mathbb{N}/\ell_\infty$.

(3)$\Rightarrow$(1). Assume for contradiction that there exists an $\alpha \in \prod_n X_n$ with $\sup_n \|\sum_{i \leq n} \alpha(i)\| < +\infty$, but $\alpha \notin \mathrm{cs}(X, (X_n))$. By Cauchy criterion, there exist an $\varepsilon_0 > 0$ and a sequence $-1 = n_0 < n_1 < \cdots < n_k < \cdots$ such that $\|\sum_{n=n_k+1}^{n_{k+1}} \alpha(n)\| \geq \varepsilon_0$. Denote $D = \{d \in \mathbb{R}^\mathbb{N} : \forall n (4^n d(n) \in \mathbb{Z})\}$. Without loss of generality, we may assume that $\alpha(n) \neq 0$ for each n, otherwise, we may replace α by $\alpha + \gamma$ for a suitable $\gamma \in \mathrm{cs}(X, (X_n))$. Define $T : D \to \prod_n X_n$ as $T(d)(n) = d(n)\alpha(n)$ for any $d \in D$ and $n \in \mathbb{N}$. It is clear that T is a homeomorphic embedding.

Suppose θ is a Borel reduction of $E(X, (X_n))$ to $\mathbb{R}^\mathbb{N}/\ell_\infty$. Then $\theta \circ T$ is also Borel on D. There exists a dense G_δ subset $G \subseteq D$ such that $\theta \circ T$ is continuous on G (cf. [15, (8.38)]). Then θ is also continuous on $T(G)$. Applying Lemma 3.1 with $a(n) = 1$ for each n, There exist $b^* \in D$ and a strictly increasing sequence (k_l) with $k_0 = 0$ such that $G \supseteq C$, where

$$C = \{d \in D : \forall l \exists i \leq 2^l \forall n \in (n_{k_l}, n_{k_{l+1}}] (d(n) = b^*(n) + i/2^l)\}.$$

Let $C^* = T(C) \cap (T(b^*) + \mathrm{cs}(X, (X_n)))$. Note that $T(C)$ is closed in $\prod_n X_n$, so C^* is relatively closed in $T(b^*) + \mathrm{cs}(X, (X_n))$. By the definition of Borel reduction, $C^* = (\theta \restriction T(C))^{-1}(\theta(T(b^*)) + \ell_\infty)$. So C^* is F_σ in $T(C)$, since θ is continuous on $T(C)$ and ℓ_∞ is F_σ in $\mathbb{R}^\mathbb{N}$. Thus we can assume that $C^* = \bigcup_m F_m$ with each F_m closed in $T(C)$.

Now denote $Z = S^{-1}(C^* - T(b^*)), Z_m = S^{-1}(F_m - T(b^*))$. Then Z is closed in X, thus complete. Because each Z_m is closed in Z with $\bigcup_m Z_m = Z$, there exists an m such that Z_m has an inner point in Z. Thus there exist $y^\# \in Z$ and $r > 0$ such that

$$Z_m \supseteq W = \{x \in Z : \|x - y^\#\| \leq r\}.$$

Let $y^\# = \sum_n y_n$ with $y_n \in X_n$, by Cauchy criterion, we have

$$\lim_{k \to \infty} \|\sum_{n=n_k+1}^{n_{k+1}} y_n\| = 0.$$

Since $S(y^\#) \in (C^* - T(b^*)) \subseteq (T(C) - T(b^*))$, we have $T^{-1}(S(y^\#)) \in (C - b^*)$. So there is a sequence (i_l) such that, for $n_{k_l} < n \leq n_{k_{l+1}}$, $T^{-1}(S(y^\#))(n) = i_l/2^l$, i.e., $y_n = S(y^\#)(n) = (i_l/2^l)\alpha(n)$. Thus

$$\lim_{l \to \infty} \frac{i_l}{2^l} \|\sum_{n=n_{k_l}+1}^{n_{k_{l+1}}} \alpha(n)\| = 0.$$

Comparing with $\|\sum_{n=n_{k_l}+1}^{n_{k_{l+1}}}\alpha(n)\|\geq\varepsilon_0$, we get $\lim_{l\to\infty} i_l/2^l=0$. Now fix a large enough natural number L such that $i_l/2^l<1/2$ for $l\geq L$, and

$$\frac{1}{2^L}\sup_n\|\sum_{i\leq n}\alpha(i)\|\leq\frac{r}{2}.$$

We define

$$\alpha'(n)=\begin{cases}0, & n\leq n_{k_L},\\ \alpha(n), & n>n_{k_L},\end{cases}$$

and for each $j>L$,

$$\alpha'_j(n)=\begin{cases}0, & n\leq n_{k_L}\text{ or }n>n_{k_j},\\ \alpha(n), & n_{k_L}<n\leq n_{k_j}.\end{cases}$$

It is clear that $T(b^*)+S(y^\#)+a'/2^L\in T(C)$ and $T(b^*)+S(y^\#)+a'_j/2^L\in C^*$ for each j. Note that $S^{-1}(\alpha'_j/2^L)=1/2^L\sum_{n=n_{k_L}+1}^{n_{k_j}}\alpha(n)$ and

$$\frac{1}{2^L}\|\sum_{n=n_{k_L}+1}^{n_{k_j}}\alpha(n)\|\leq\frac{1}{2^L}\left(\|\sum_{i\leq n_{k_L}}\alpha(i)\|+\|\sum_{i\leq n_{k_j}}\alpha(i)\|\right)\leq r.$$

It follows that $y^\#+S^{-1}(\alpha'_j/2^L)\in W\subseteq Z_m=S^{-1}(F_m-T(b^*))$, i.e., $T(b^*)+S(y^\#)+\alpha'_j/2^L\in F_m$. Since F_m is closed in $T(C)$, we have

$$T(b^*)+S(y^\#)+\alpha'/2^L=\lim_{j\to\infty}(T(b^*)+S(y^\#)+\alpha'_j/2^L)\in F_m\subseteq C^*.$$

From the definition of C^*, we have $S(y^\#)+\alpha'/2^L\in\mathrm{cs}(X,(X_n))$, so $\alpha'\in\mathrm{cs}(X,(X_n))$. Hence $\alpha\in\mathrm{cs}(X,(X_n))$, a contradiction! $\square$

Indeed, the proof of (3)$\Rightarrow$(1) shows that, if (X_n) is not boundedly complete, then $E(X,(X_n))$ is not Borel reducible to any F_σ equivalence relation.

Theorem 1.1 is equivalent to the following corollary.

Corollary 3.3. *Let (x_n) be a Schauder basis of a Banach space X. Then the following are equivalent:*

(1) *(x_n) is boundedly complete;*
(2) $\mathrm{coef}(X,(x_n))$ *is F_σ in $\mathbb{R}^\mathbb{N}$;*
(3) $E(X,(x_n))\leq_B\mathbb{R}^\mathbb{N}/\ell_\infty$.

4. One lemma on additive reductions

A lemma for converting a Borel reduction to an additive reduction will be used again and again in the rest of this paper, especially for proving nonreducibility. We introduce some concerned notions first.

Definition 4.1 (Farah [6]). (a) A map $\psi:\prod_n X_n\to\prod_n X'_n$ is *additive* if there are $0=l_0<l_1<\cdots<l_j<\cdots$ and maps $H_j:X_j\to\prod_{n\in[l_i,l_{i+1})}X'_i$ such that

$$\psi(\alpha)=H_0(\alpha(0))^\frown H_1(\alpha(1))^\frown H_2(\alpha(2))^\frown\cdots.$$

(b) Let E and F be equivalence relations on $\prod_n X_n$ and $\prod_n X'_n$ respectively, we say E is *additively reducible* to F, denoted $E \leq_A F$, if there is an additive reduction of E to F.

Let E be an equivalence relation on $\prod_n X_n$, and let $I \subseteq \mathbb{N}$ be infinite. Fix an element $\mu \in \prod_{n \notin I} X_n$. For $\alpha \in \prod_{n \in I} X_n$, put $\alpha \oplus \mu = \begin{cases} \alpha(n), & n \in I, \\ \mu(n), & n \notin I. \end{cases}$ We define $E|_I$ (with μ) on $\prod_{n \in I} X_n$ as

$$(\alpha, \beta) \in E|_I \iff (\alpha \oplus \mu, \beta \oplus \mu) \in E.$$

If for any $\alpha_1, \alpha_2 \in \prod_n X_n$, $(\alpha_1, \alpha_2) \in E$ is equivalent to say $\alpha_1 - \alpha_2$ is in a specified set, then the exact value of $\mu(n)$ is meaningless, thus we may assume, say, $\mu(n) = 0$ for all $n \notin I$, if we need.

Let (F_n) be a sequence of finite sets. A special equivalence relation $E_0(\prod_n F_n)$ defined as

$$(\alpha, \beta) \in E_0(\prod_n F_n) \iff \exists m \forall n > m(\alpha(n) = \beta(n))$$

for all $\alpha, \beta \in \prod_n F_n$.

A weak version of the following lemma is due to Dougherty and Hjorth [4].

Lemma 4.2. *Let E be an equivalence relation on $\prod_n F_n$ with $E_0(\prod_n F_n) \subseteq E$, where all F_n are finite sets. Let (X_n) be a Schauder decomposition of a separable Banach space X. If $E \leq_B E(X, (X_n))$, then there exists an infinite $I \subseteq \mathbb{N}$ such that $E|_I \leq_A E(X, (X_n))$.*

Proof. The proof is modified from the proof of [4], Theorem 2.2, claims (i)–(iii). We omit some similar arguments.

Assume that θ is a Borel reduction of E to $E(X, (X_n))$. Following claims (i) and (ii), and the arguments after Claim (ii) of [4], Theorem 2.2, we construct two sequences of natural numbers $n_0 < n_1 < n_2 < \cdots$ and $l_0 < l_1 < l_2 < \cdots$, a sequence of (s_j), and dense open sets $D_i^j \subseteq \prod_n F_n$ $(i, j \in \mathbb{N})$.

Denote $I = \{n_j : j \in \mathbb{N}\}$ and $\mu = \bigcup_j s_j$. Note that $\mathrm{dom}(\mu) = \bigcup_j \mathrm{dom}(s_j) = \mathbb{N} \setminus I$. The construction confirms that, for any $\alpha, \hat{\alpha} \in \prod_{n \in I} F_n$, we have

(a) if $\alpha(n) = \hat{\alpha}(n)$ for $n > n_j$, then

$$\|\sum_{n \geq l_{j+1}} (\theta(\alpha \oplus \mu)(n) - \theta(\hat{\alpha} \oplus \mu)(n))\| < 2^{-j};$$

(b) if $\alpha(n) = \hat{\alpha}(n)$ for $n \leq n_j$, then

$$\|\sum_{n < l_{j+1}} (\theta(\alpha \oplus \mu)(n) - \theta(\hat{\alpha} \oplus \mu)(n))\| < 2^{-j}.$$

Now we are ready to define a sequence of mappings $(H_n)_{n \in I}$ to assemble the desired additive reduction ψ. For each $i \in I$, fix an element $x_n^* \in F_n$. We define $p_j : F_{n_j} \to \prod_{n \in I} F_n$ for each $j \in \mathbb{N}$ as $p_j(x)(n) = \begin{cases} x, & n = n_j, \\ x_n^*, & n \neq n_j \end{cases}$

for $x \in F_n$ and $n \in \mathbb{N}$. Then for $n \in I$ with $n = n_j$, we define $H_n : F_n \to \prod_{n\in[l_j,l_{j+1})} X_n$ as, for $x \in F_n$,

$$H_n(x) = \theta(p_j(x) \oplus \mu) \restriction [l_j, l_{j+1}).$$

The additive mapping $\psi : \prod_{n\in I} F_n \to \prod_n X_n$ defined as, for $\alpha \in \prod_{n\in I} F_n$,

$$\psi(\alpha) = H_{n_0}(\alpha(n_0))^\frown H_{n_1}(\alpha(n_1))^\frown H_{n_2}(\alpha(n_2))^\frown \cdots .$$

We come to show that ψ is a reduction of $E|_I$ to $E(X, (X_n))$.

For any $\alpha \in \prod_{n\in I} F_n$ and $j \in \mathbb{N}$, define $e_j(\alpha), e'_j(\alpha) \in \prod_{n\in I} F_n$ as

$$e_j(\alpha)(n) = \begin{cases} \alpha(n), & n = n_j, \\ x_n^*, & n \neq n_j, \end{cases} \qquad e'_j(\alpha)(n) = \begin{cases} \alpha(n), & n \leq n_j, \\ x_n^*, & n > n_j. \end{cases}$$

Applying (a) for $j-1$ and (b) for j, we have

$$\|\sum_{n\geq l_j}(\theta(e_j(\alpha) \oplus \mu)(n) - \theta(e'_j(\alpha) \oplus \mu)(n))\| < 2^{-(j-1)},$$

$$\|\sum_{n<l_{j+1}}(\theta(\alpha \oplus \mu)(n) - \theta(e'_j(\alpha) \oplus \mu)(n))\| < 2^{-j}.$$

Let M be the decompositon constant of (X_n). We have

$$\begin{array}{rl} & \|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha \oplus \mu)(n) - \theta(e_j(\alpha) \oplus \mu)(n))\| \\ \leq & \|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha \oplus \mu)(n) - \theta(e'_j(\alpha) \oplus \mu)(n))\| \\ & +\|\sum_{n\in[l_j,l_{j+1})}(\theta(e'_j(\alpha) \oplus \mu)(n)) - \theta(e_j(\alpha) \oplus \mu)(n))\| \\ \leq & (1+M)\|\sum_{n<l_{j+1}}(\theta(\alpha \oplus \mu)(n) - \theta(e'_j(\alpha) \oplus \mu)(n))\| \\ & +M\|\sum_{n\geq l_j}(\theta(e'_j(\alpha) \oplus \mu)(n)) - \theta(e_j(\alpha) \oplus \mu)(n))\| \\ < & (1+3M)2^{-j}. \end{array}$$

Note that $p_j(\alpha(n_j)) = e_j(\alpha)$. We have

$$\psi(\alpha) \restriction [l_j, l_{j+1}) = H_{n_j}(\alpha(n_j)) = \theta(e_j(\alpha) \oplus \mu) \restriction [l_j, l_{j+1})$$

for each $j \in \mathbb{N}$. For each $m \in \mathbb{N}$, let $n_i \leq m < n_{i+1}$. We have

$$\begin{array}{rl} & \|\sum_{n\geq m}(\theta(\alpha \oplus \mu)(n) - \psi(\alpha)(n))\| \\ \leq & \|\sum_{n\in[m,l_{i+1})}(\theta(\alpha \oplus \mu)(n) - \theta(e_i(\alpha) \oplus \mu)(n))\| \\ & +\sum_{j\geq i+1}\|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha \oplus \mu)(n) - \theta(e_j(\alpha) \oplus \mu)(n))\| \\ < & (1+M)(1+3M)2^{-i} + \sum_{j\geq i+1}(1+3M)2^{-j} \\ = & (2+M)(1+3M)2^{-i}. \end{array}$$

It follows that

$$\lim_{m\to\infty}\sum_{n\geq m}(\theta(\alpha \oplus \mu)(n) - \psi(\alpha)(n)) = 0,$$

i.e., $(\theta(\alpha \oplus \mu), \psi(\alpha)) \in E(X, (X_n))$.

In the end, for $\alpha, \hat{\alpha} \in \prod_{n\in I} F_n$, we have

$$\begin{array}{rcl} (\psi(\alpha), \psi(\hat{\alpha})) \in E(X, (X_n)) & \iff & (\theta(\alpha \oplus \mu), \theta(\hat{\alpha} \oplus \mu)) \in E(X, (X_n)) \\ & \iff & (\alpha \oplus \mu, \hat{\alpha} \oplus \mu) \in E \\ & \iff & (\alpha, \hat{\alpha}) \in E|_I. \end{array}$$

This completes the proof. $\square$

Remark 4.3. It worth noting that the preceding lemma can be applied on some variations. If there is a subset $M_n \subseteq X_n$ for each n such that $E \leq_B E(X,(X_n)) \upharpoonright \prod_n M_n$, i.e., there is a Borel reduction θ from $\prod_n F_n$ to $\prod_n M_n$, then the resulted additive reduction ψ is also mapping into $\prod_n M_n$. Thus $E|_I \leq_A E(X,(X_n)) \upharpoonright \prod_n M_n$.

As an application, we prove the following theorem.

Theorem 4.4. *Let (x_n) and (y_n) be bases of Banach spaces X and Y respectively. If (x_n) is unconditional, and every subsequence of (y_n) is conditional, then $E(Y,(y_n)) \not\leq_B E(X,(x_n))$.*

Proof. Assume for contradiction that $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a Borel reduction of $E(Y,(y_n))$ to $E(X,(x_n))$. Denote $F_n = \{i/2^n : i = 0, 1, \cdots, 2^n\}$, and denote by E the restriction of $E(Y,(y_n))$ on $\prod_n F_n$. Then $\theta \upharpoonright \prod_n F_n$ is also a Borel reduction of E to $E(X,(x_n))$. From Lemma 4.2, we can find an infinite set $I \subseteq \mathbb{N}$ and an element $\mu \in \prod_{n\notin I} F_n$ such that $E|_I$ (with μ) is additively reducible to $E(X,(x_n))$. Without loss of generality, we may assume that $\mu(n) = 0$ for $n \notin I$. Therefore, we can find, for each $n \in I$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to \mathbb{R}^{l_n}$ such that the following ψ is a reduction of $E|_I$ to $E(X,(x_n))$. Let (n_k) is the strictly increasing enumeration of I, then ψ is defined as

$$\psi(a) = H_{n_0}(a(n_0))^\frown H_{n_1}(a(n_1))^\frown H_{n_2}(a(n_2))^\frown \cdots,$$

for any $a \in \prod_{n\in I} F_n$.

From the assumption of (y_n), the subsequence (y_{n_k}) is conditional. By Definition 1.c.5 and Proposition 1.c.6 of [16], there exists an $a_0 \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_k a_0(k) y_{n_k}$ converges conditionally. Then by Proposition 1.c.1 of [16], there is an $\epsilon \in \{-1,1\}^{\mathbb{N}}$ such that $\sum_k a_0(k) y_{n_k}$ converges while $\sum_k \epsilon(k) a_0(k) y_{n_k}$ diverges.

Without loss of generality, we may assume that (y_n) is normalized. Then we have $\ell_1 \subseteq \mathrm{coef}(Y,(y_{n_k})) \subseteq c_0$, so we may further assume that $a_0(k) \in F_{n_k}$. For each $n_k \in I$, denote $a^I(n_k) = a_0(k), a^\emptyset(n) = 0$, and define

$$a^+(n_k) = \begin{cases} a_0(k), & \epsilon(k) = 1, \\ 0, & \epsilon(k) = -1, \end{cases} \qquad a^-(n_k) = \begin{cases} 0, & \epsilon(k) = 1, \\ a_0(k), & \epsilon(k) = -1. \end{cases}$$

Since $\sum_{n\in I}(a^I(n) - a^\emptyset(n)) y_n$ converges and $\sum_{n\in I}(a^+(n) - a^-(n)) y_n$ diverges, we have $\psi(a^I) - \psi(a^\emptyset) \in \mathrm{coef}(X,(x_n))$ and $\psi(a^+) - \psi(a^-) \notin \mathrm{coef}(X,(x_n))$. Denote $t_k = H_{n_k}(a_0(k)) - H_{n_k}(0)$. Then we have

$$t_0{}^\frown t_1{}^\frown t_2{}^\frown \cdots \in \mathrm{coef}(X,(x_n)),$$

$$\epsilon(0)t_0{}^\frown \epsilon(1)t_1{}^\frown \epsilon(2)t_2{}^\frown \cdots \notin \mathrm{coef}(X,(x_n)).$$

This contradicts the unconditionality of (x_n) (cf. Proposition 1.c.6 of [16]). $\square$

5. Different Schauder bases of a Banach space

A question arises naturally:

> If (x_n) and (y_n) are two Schauder bases of a Banach space X, does $E(X,(x_n)) \sim_B E(X,(y_n))$?

Recall that two sequences (x_n) and (y_n) of X are equivalent if $\mathrm{coef}(X,(x_n)) = \mathrm{coef}(X,(y_n))$. It is well known that, in every infinite dimensional Banach space with a basis, there exist continuum many mutually non-equivalent normalized bases (see [19], 4.1). If we restrict on unconditional bases, only c_0, ℓ_1 and ℓ_2 have the property that all unconditional bases are equivalent (see [16], Theorem 2.b.10).

When we return to Borel bireducibility between $E(X,(x_n))$ and $E(X,(y_n))$, the question becomes very complicated. From Corollary 3.3, if (x_n) is boundedly complete and (y_n) is not, then $E(X,(x_n)) \not\sim_B E(X,(y_n))$. In [23], Zippin proved that, for a Banach space X with a basis, X is reflexive iff all bases of X are boundedly complete. So if a non-reflexive space X has a boundedly complete basis, the question turns out to fail for X. Which spaces the question can hold for? Perhaps, the most possible candidates might be reflexive spaces. So far, the only known example of such space is Hilbert space ℓ_2.

Lemma 5.1. *Let (x_n) and (y_k) be two bases of a Banach space X. If $y_k = \sum_n \alpha_{nk} x_n$ with $\alpha_{nk} = 0$ for any $n < k$, then $E(X,(x_n)) \sim_B E(X,(y_k))$.*

Proof. For any $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $\theta(a)(n) = \sum_{k \le n} \alpha_{nk} a(k)$. Since (y_k) is a basis of X, we can assume that $x_n = \sum_k \beta_{kn} y_k$ for each n. Then

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \ne n \end{cases} = x_m^*(x_n) = \sum_k \beta_{kn} x_m^*(y_k) = \sum_{k \le m} \beta_{kn} \alpha_{mk}.$$

By induction on m, we can prove that $\beta_{kn} = 0$ for any $k < n$. Furthermore, we also have $\sum_{n \le m} \alpha_{nk} \beta_{mn} = y_m^*(y_k) = \delta_{mk}$. Therefore, $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is invertible and $\theta^{-1}(d)(k) = \sum_{n \le k} \beta_{kn} d(n)$ for any $d \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$.

Let $a, b \in \mathbb{R}^{\mathbb{N}}$. If $a - b \in \mathrm{coef}(X,(y_k))$, then there is $x \in X$ such that $x = \sum_k (a(k) - b(k)) y_k$. Note that

$$x_n^*(x) = \sum_k (a(k) - b(k)) x_n^*(y_k) = \sum_{k \le n} \alpha_{nk}(a(k) - b(k)).$$

We have $x = \sum_n x_n^*(x) x_n = \sum_n (\theta(a)(n) - \theta(b)(n)) x_n$. Thus $\theta(a) - \theta(b) \in \mathrm{coef}(X,(x_n))$. On the other hand, using θ^{-1}, we can prove that, if $\theta(a) - \theta(b) \in \mathrm{coef}(X,(x_n))$, then $a - b \in \mathrm{coef}(X,(y_k))$.

Therefore, θ and θ^{-1} witness $E(X,(x_n)) \sim_B E(X,(y_k))$. □

Theorem 5.2. *For any basis (y_k) of ℓ_2, we have $E(\ell_2,(y_k)) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_2$.*

Proof. For each $n \in \mathbb{N}$, denote $X_n = [y_k]_{k \ge n}$. Find a normalized $x_n \in X_n$ such that $x_n \perp X_{n+1}$. Then we have $\{x_0, \cdots, x_n\}^\perp = X_{n+1}$. We claim that (x_n) is an orthonormal basis of ℓ_2. It is easy to see that $x_m \perp x_n$ for

$m \neq n$. Thus (x_n) is an orthonormal basis of $[x_n]_{n\in\mathbb{N}}$. Let $x \perp [x_n]_{n\in\mathbb{N}}$. Then $x \in \bigcap_n X_n$. Since (y_k) is a basis, let $x = \sum_k r_k y_k$. We can see that $r_k = 0$ for any k, so $x = 0$. It follows that $[x_n]_{n\in\mathbb{N}} = \ell_2$.

Let $\langle\cdot,\cdot\rangle$ be the inner product of ℓ_2. For any $n, k \in \mathbb{N}$, denote $\alpha_{nk} = \langle y_k, x_n\rangle$. Since $y_k \in X_k$, we have $\alpha_{nk} = 0$ for $n < k$. By Lemma 5.1, we have $E(\ell_2,(y_k)) \sim_B E(\ell_2,(x_n)) = \mathbb{R}^{\mathbb{N}}/\ell_2$. □

Besides Hilbert space ℓ_2, we would like to investigate the Borel reducibility between $E(X,(x_n))$'s with (x_n) a basis of X. In this section, we focus on two special spaces: c_0 and ℓ_1. Both of them are non-reflexive. ℓ_1 has boundedly complete bases, while c_0 has none.

Theorem 5.3. *Let (x_n) be a basis of a Banach space X. Let $y_k = \sum_n \alpha_{nk} x_n$ satisfies that, for any n, there are only finitely many k such that $\alpha_{nk} \neq 0$. Then*

$$E(X,(y_k)) \leq_B E(X,(x_n)) \otimes \mathbb{R}^{\mathbb{N}}/c_0.$$

Proof. For $m \in \mathbb{N}$, denote $N_m = \max\{N : \forall n \leq N \forall k > m(\alpha_{nk} = 0)\}$. From the assumption of α_{nk}, we can see $\lim_{m\to\infty} N_m = \infty$. Define $\theta_1 : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\theta_1(a)(n) = \sum_k a(k)\alpha_{nk}$ for $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, and define $\theta_2 : \mathbb{R}^{\mathbb{N}} \to X^{\mathbb{N}}$ by

$$\theta_2(a)(m) = \sum_{k\leq m} a(k) \sum_{n>N_m} \alpha_{nk} x_n$$

for $a \in \mathbb{R}^{\mathbb{N}}$ and $m \in \mathbb{N}$. The definition of N_m implies that, for $n \leq N_m$, $\theta_1(a)(n) = \sum_{k\leq m} a(k)\alpha_{nk}$.

Let $a, b \in \mathbb{R}^{\mathbb{N}}$. We have

$$\begin{aligned}
& \textstyle\sum_{k\leq m}(a(k)-b(k))y_k \\
= & \textstyle\sum_{k\leq m}(a(k)-b(k))\left(\sum_{n\leq N_m}\alpha_{nk}x_n + \sum_{n>N_m}\alpha_{nk}x_n\right) \\
= & \textstyle\sum_{n\leq N_m}\left(\sum_{k\leq m}(a(k)-b(k))\alpha_{nk}\right)x_n + (\theta_2(a)(m)-\theta_2(b)(m)) \\
= & \textstyle\sum_{n\leq N_m}(\theta_1(a)(n)-\theta_1(b)(n))\,x_n + (\theta_2(a)(m)-\theta_2(b)(m)).
\end{aligned}$$

If $a - b \in \mathrm{coef}(X,(y_k))$, we denote $x = \sum_k (a(k)-b(k))y_k$. Since (x_n) is a basis, we have

$$\begin{aligned}
x &= \textstyle\sum_n x_n^*(x)x_n \\
&= \textstyle\sum_n \left(\sum_k (a(k)-b(k))x_n^*(y_k)\right)x_n \\
&= \textstyle\sum_n \left(\sum_k (a(k)-b(k))\alpha_{nk}\right)x_n \\
&= \textstyle\sum_n (\theta_1(a)(n)-\theta_1(b)(n))x_n.
\end{aligned}$$

Thus $\theta_1(a) - \theta_1(b) \in \mathrm{coef}(X,(x_n))$. Furthermore, by $\lim_{m\to\infty} N_m = \infty$, we have

$$x = \lim_{m\to\infty}\sum_{k\leq m}(a(k)-b(k))y_k = \lim_{m\to\infty}\sum_{n\leq N_m}(\theta_1(a)(n)-\theta_1(b)(n))x_n.$$

Therefore $\lim_{m\to\infty}\|\theta_2(a)(m)-\theta_2(b)(m)\| = 0$.

On the other hand, assume that

$$\theta_1(a)-\theta_1(b)\in\mathrm{coef}(X,(x_n)) \quad \& \quad \lim_{m\to\infty}\|\theta_2(a)(m)-\theta_2(b)(m)\|=0.$$

Then $\sum_n(\theta_1(a)(n)-\theta_1(b)(n))\,x_n$ is convergent. Furthermore, we have

$$\begin{aligned}\textstyle\sum_k(a(k)-b(k))y_k &= \textstyle\lim_{m\to\infty}\sum_{k\le m}(a(k)-b(k))y_k\\ &= \textstyle\lim_{m\to\infty}\sum_{n\le N_m}(\theta_1(a)(n)-\theta_1(b)(n))x_n\\ &= \textstyle\sum_n(\theta_1(a)(n)-\theta_1(b)(n))x_n.\end{aligned}$$

It follows that $a-b\in\mathrm{coef}(X,(y_k))$.

By Theorem 3.4 of [3], there is a Borel function $\theta':X^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$ such that, for $x,y\in X^{\mathbb{N}}$,

$$\lim_{m\to\infty}\|x(m)-y(m)\|=0 \iff (\theta'(x)-\theta'(y))\in c_0.$$

Now we can define the desired Borel reduction of $E(X,(y_k))$ to $E(X,(x_n))\otimes\mathbb{R}^{\mathbb{N}}/c_0$ as $\theta(a)=(\theta_1(a),\theta'(\theta_2(a)))$ for all $a\in\mathbb{R}^{\mathbb{N}}$. □

Recall that a basis (x_n) of a Banach space X is called *subsymmetric* if it is unconditional and any subsequence (x_{n_k}) of (x_n) is equivalent to (x_n) itself. The unit vector bases of c_0 and ℓ_p are subsymmetric. The following theorem is due to Xin Ma.

Theorem 5.4 ([17], Theorem 1.1)**.** *Let (y_n) be a basis of a Banach space Y, and (x_n) a subsymmetric basis of a closed subspace X of Y. Then $E(X,(x_n))\le_B E(Y,(y_n))$.*

5.1. **Bases in c_0.** By Theorem 5.4, among all $E(c_0,(x_n))$ with (x_n) a basis of c_0, $\mathbb{R}^{\mathbb{N}}/c_0$ is the minimum element with respect to Borel reducibility. We are going to find a maximum among them.

We denote

$$\mathrm{cs}=\{a\in\mathbb{R}^{\mathbb{N}}:\sum_n a(n)\text{ converges}\}.$$

Let $x_n^1=\sum_{k\le n}e_k$, where (e_k) is the unit vector basis of c_0. Then (x_n^1) is a basis of c_0 too. Since

$$\sum_n a(n)x_n^1=(\sum_{n\ge0}a(n),\sum_{n\ge1}a(n),\cdots,\sum_{n\ge k}a(n),\cdots),$$

we can see $\mathrm{cs}=\mathrm{coef}(c_0,(x_n^1))$.

A simple reduction $\theta(a)=(a(0),-a(0),a(1),-a(1),\cdots)$ witnesses that $\mathbb{R}^{\mathbb{N}}/c_0\le_B\mathbb{R}^{\mathbb{N}}/\mathrm{cs}$. We can easily prove $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}\sim_B\mathbb{R}^{\mathbb{N}}/c$, where c is the set of all convergent sequences. It is worth noting that the unit vectors cannot form a basis of c.

For $m\ge1$, note that $\bigoplus_{i=1}^m c_0\cong c_0$. We choose a suitable basis (x_n^m) of c_0 and define $\mathrm{cs}^{(m)}=\mathrm{coef}(c_0,(x_n^m))$. To do so, let (e_k^i) be the unit vector basis of the i-th component space c_0, then we set $x_{mj+i-1}^m=\sum_{k\le j}e_k^i$ for $i\le m$ and $j\in\mathbb{N}$.

Furthermore, recall that

$$\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0 = \{(a_n)\in (c_0)^{\mathbb{N}} : \lim_{n\to\infty}\|a_n\| = 0\}.$$

We still have $\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0 \cong c_0$. Now fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2\to\mathbb{N}$ such that, for any i, $\langle i,j\rangle$ is strictly increasing with respect to variable j. Let (e^i_k) be the unit vector basis of the i-th component space c_0 in $\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0$. We set $x^\infty_n = \sum_{k\le j} e^i_k$ for $n = \langle i,j\rangle$. We can see that (x^∞_n) is also a basis of c_0. Now we denote

$$\mathrm{cs}^{(\infty)} = \mathrm{coef}(c_0, (x^\infty_n)).$$

It is straight forward to check that

$$\mathrm{cs}^{(m)} = \{a\in\mathbb{R}^{\mathbb{N}} : \forall i\le m(\sum_j a(mj+i-1) \text{ converges})\},$$

$$\mathrm{cs}^{(\infty)} = \{a\in\mathbb{R}^{\mathbb{N}} : \forall j(a(\langle\cdot,j\rangle)\in c_0)\ \&\ \sum_j a(\langle\cdot,j\rangle) \text{ converges in } c_0\}.$$

For any Banach space X, we define

$$\mathrm{cs}(X) = \{\alpha\in X^{\mathbb{N}} : \sum_n \alpha(n) \text{ converges in } X\},$$

and for any $\alpha\in\mathrm{cs}(X)$, define

$$\|\alpha\|_X = \sup_n \|\sum_{i\le n}\alpha(i)\|.$$

By Cauchy criterion, it is straightforward to check that $(\mathrm{cs}(X), \|\cdot\|_X)$ is a Banach space. Furthermore, letting $X_n = \{\alpha\in\mathrm{cs}(X) : \forall i\ne n(\alpha(i)=0)\}$, we can see that (X_n) forms a Schauder decomposition of $\mathrm{cs}(X)$. For any sequence (x_n) in X, we claim that

$$E(X,(x_n)) \le_B X^{\mathbb{N}}/\mathrm{cs}(X).$$

Indeed, for any $a\in\mathbb{R}^{\mathbb{N}}$ and $k\in\mathbb{N}$, we define $\theta(a)(k) = a(k)x_k$. Then θ is a Borel reduction of $E(X,(x_n))$ to $X^{\mathbb{N}}/\mathrm{cs}(X)$.

An easy observation shows that

$$(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)},$$

$$(c_0)^{\mathbb{N}}/\mathrm{cs}(c_0) \le_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)} = E(c_0,(x^{(\infty)}_n)) \le_B (c_0)^{\mathbb{N}}/\mathrm{cs}(c_0).$$

Therefore, $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$ is the desired maximum element. Furthermore, we have the following Theorem.

Theorem 5.5. *Let (x_n) be a none-zero sequence in c_0. Then*

$$\mathbb{R}^{\mathbb{N}}/c_0 \le_B E(c_0,(x_n)) \le_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$

Proof. $E(c_0,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$ follows from $(c_0)^{\mathbb{N}}/\mathrm{cs}(c_0) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$. We only need to prove $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0,(x_n))$.

We may assume that (x_n) is normalized. Note that $c_0^{**} = \ell_\infty$ and the unit ball of ℓ_∞ is weak* compact. There is a subsequence of (x_n) which is weak* convergent in ℓ_∞. Without loss of generality, assume that (x_n) itself is weak* convergent.

Case 1. If (x_n) is not convergent in c_0, by the Eberlein-Šmulian theorem (cf. [2, p. 41]), there is a subsequence (x_{n_k}) of (x_n) which is a basic sequence. By Proposition 1.a.11 of [16], there is a basic sequence (y_n) in $[x_{n_k}]_{k\in\mathbb{N}}$ which is equivalent to a block basis (u_j) of the unit vector basis (e_n). We may also assume that (u_j) is normalized. From Proposition 2.a.1 of [16], (u_j) is equivalent to (e_n). Let $(x^*_{n_k})$ be the biorthogonal functionals in $[x_{n_k}]^*_{k\in\mathbb{N}}$. Then $x^*_{n_k} \upharpoonright [y_n]_{n\in\mathbb{N}} \in [y_n]^*_{n\in\mathbb{N}}$. Since (y_n) is equivalent to (e_n), we have (y^*_n) is equivalent to the unit vector basis of ℓ_1. It follows that $\lim_{n\to\infty} x^*_{n_k}(y_n) = 0$ for each $k \in \mathbb{N}$. Then by Proposition 1.a.12 of [16], there is a subsequence (y_{n_j}) of (y_n) which is equivalent to a block basis of (x_{n_k}). Note that (y_{n_j}) is still equivalent to the unit vector basis of c_0, we have

$$\mathbb{R}^{\mathbb{N}}/c_0 = E(c_0,(y_{n_j})) \leq_B E(c_0,(x_{n_k})) \leq_B E(c_0,(x_n)).$$

Case 2. If (x_n) converges to $x \in c_0$. Then $\|x\| = 1$, since (x_n) is normalized. There is a subsequence (x_{n_k}) such that $\|x_{n_k} - x\| \leq 2^{-k}$. Then for any $a \in \mathbb{R}^{\mathbb{N}}$, we have

$$\sum_k a(k)x_{n_k} \text{ converges} \iff \sum_k a(k)x \text{ converges} \iff a \in \mathrm{cs}.$$

Thus $\mathbb{R}^{\mathbb{N}}/\mathrm{cs} \leq_B E(c_0,(x_n))$, and hence $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0,(x_n))$. □

Remark 5.6. Following the proof of the last theorem, we can also get: for any none-zero sequence (x_n) in ℓ_p with $p > 1$, we have

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E(\ell_p,(x_n)) \quad \text{or} \quad \mathbb{R}^{\mathbb{N}}/\mathrm{cs} \leq_B E(\ell_p,(x_n)).$$

Corollary 5.7. *Let (x_k) be a none-zero sequence in c_0. If for any n there are only finitely many k such that $x_k(n) \neq 0$, then $E(c_0,(x_k)) \sim_B \mathbb{R}^{\mathbb{N}}/c_0$.*

Proof. The last theorem implies $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0,(x_k))$. From Theorem 5.3, we get $E(c_0,(x_k)) \leq_B \mathbb{R}^{\mathbb{N}}/c_0 \otimes \mathbb{R}^{\mathbb{N}}/c_0$. So we have $E(c_0,(x_k)) \leq_B \mathbb{R}^{\mathbb{N}}/c_0$, since $\mathbb{R}^{\mathbb{N}}/c_0 \otimes \mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/c_0$ is trivial. □

Now we are going to compare $\mathbb{R}^{\mathbb{N}}/c_0$, $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$, and $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

Recall that a series $\sum_k x_k$ is said to be *perfectly divergent* if for any $\epsilon \in \{-1,1\}^{\mathbb{N}}$ the series $\sum_k \epsilon(k)x_k$ diverges. The only interesting case is when $\lim_{k\to\infty} x_k = 0$. One example in ℓ_p is $\sum_n (n+1)^{-1/p} e_n$. Another example

in c_0 is the follows:

$$\begin{array}{llrrrrrrrrrl} x_0 & =(1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots), \\ x_1 & =(0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots), \\ x_2 & =(0, & \frac{1}{2}, & -\frac{1}{2}, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots), \\ x_3 & =(0, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & \cdots), \\ x_4 & =(0, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & -\frac{1}{3}, & -\frac{1}{3}, & 0, & 0, & \cdots), \\ x_5 & =(0, & 0, & 0, & \frac{1}{3}, & -\frac{1}{3}, & \frac{1}{3}, & -\frac{1}{3}, & 0, & 0, & \cdots), \\ \cdots & \cdots \end{array}$$

In fact, Dvoretzky proved that, in any infinite-dimensional Banach space, there are perfectly divergent series whose general term tends to 0 (see, e.g. [13], Theorem 6.2.1). On the other hand, Dvoretzky-Hanani's Theorem shows that for any perfectly divergent series in a finite-dimensional space, its general term does not tend to 0 (see, e.g. [13], Theorem 2.2.1).

Lemma 5.8. *Let X be a separable infinite-dimensional Banach space and Y a finite-dimensional normed space. Then*

$$X^{\mathbb{N}}/\mathrm{cs}(X) \not\leq_B Y^{\mathbb{N}}/\mathrm{cs}(Y).$$

Proof. Let $\sum_k x_k$ be a perfectly divergent series in X with $\lim_{k\to\infty} x_k = 0$. We denote $F_n = \{0\} \cup \{x_k : k \leq n\}$ for each $n \in \mathbb{N}$, and denote by E the restriction of $X^{\mathbb{N}}/\mathrm{cs}(X)$ on $\prod_n F_n$. Assume for contraction that $X^{\mathbb{N}}/\mathrm{cs}(X) \leq_B Y^{\mathbb{N}}/\mathrm{cs}(Y)$. Then we also have $E \leq_B Y^{\mathbb{N}}/\mathrm{cs}(Y)$.

From Lemma 4.2, we can find an infinite set $I \subseteq \mathbb{N}$ and an element $\mu \in \prod_{n\notin I} F_n$ such that $E|_I$ (with μ) is additively reducible to $Y^{\mathbb{N}}/\mathrm{cs}(Y)$. Without loss of generality, we may assume that $\mu(n) = 0$ for $n \notin I$. Therefore, we can find, for each $n \in I$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to Y^{l_n}$ such that the following ψ is a reduction of $E|_I$ to $Y^{\mathbb{N}}$. Let (n_k) is the strictly increasing enumeration of I, then ψ is defined as

$$\psi(\alpha) = H_{n_0}(\alpha(n_0))^\frown H_{n_1}(\alpha(n_1))^\frown H_{n_2}(\alpha(n_2))^\frown \cdots,$$

for any $\alpha \in \prod_{n\in I} F_n$.

For $(y_0, \cdots, y_{l-1}) \in Y^l$, denote

$$|\!|\!|(y_0, \cdots, y_{l-1})|\!|\!|_Y = \max_{i<l} \|y_0 + \cdots + y_i\|.$$

We claim that $\lim_{k\to\infty} |\!|\!|H_{n_k}(x_k) - H_{n_k}(0)|\!|\!|_Y = 0$. If not, there shall be an infinite $K \subseteq \mathbb{N}$ and $\varepsilon > 0$ such that $|\!|\!|H_{n_k}(x_k) - H_{n_k}(0)|\!|\!|_Y \geq \varepsilon$ for each $k \in K$. Since $\lim_{k\to\infty} x_k = 0$, we can find an infinite $J \subseteq K$ with $\sum_{k\in J} x_k$ converging. Now define $\alpha^\emptyset(n) = 0$ for each $n \in I$ and $\alpha^J \in \prod_{n\in I} F_n$ as

$$\alpha^J(n) = \begin{cases} x_k, & n = n_k, k \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n\in I}(\alpha^J(n) - \alpha^\emptyset(n)) = \sum_{k\in J} x_k$ converges, so $(\alpha^J, \alpha^\emptyset) \in E|_I$. Therefore $\psi(\alpha^J) - \psi(\alpha^\emptyset) \in \mathrm{cs}(Y)$. By Cauchy criterion, we have

$$\lim_{k\to\infty} |\!|\!|H_{n_k}(\alpha^J(n_k)) - H_{n_k}(\alpha^\emptyset(n_k))|\!|\!|_Y = 0.$$

This contradicts with $J \subseteq K$.

By Dvoretzky-Hanani's Theorem, $\sum_k (H_{n_k}(x_k) - H_{n_k}(0))$ is not perfectly divergent, so there is a sequence $\epsilon \in \{-1,1\}^{\mathbb{N}}$ such that

$$\sum_k \epsilon(k)(H_{n_k}(x_k) - H_{n_k}(0)) \text{ converges.}$$

Now define $\alpha^+, \alpha^- \in \prod_{n\in I} F_n$ as

$$\alpha^+(n_k) = \begin{cases} x_k, & \epsilon(k) = 1, \\ 0, & \epsilon(k) = -1, \end{cases} \qquad \alpha^-(n_k) = \begin{cases} 0, & \epsilon(k) = 1, \\ x_k, & \epsilon(k) = -1. \end{cases}$$

We have $\sum_{n\in I}(\alpha^+(n) - \alpha^-(n)) = \sum_k \epsilon(k)x_k$ diverges, since $\sum_k x_k$ is perfectly divergent. It follows that $(\psi(\alpha^+) - \psi(\alpha^-)) \notin \mathrm{cs}(Y)$. On the other hand, we can see that

$$\sum_k (H_{n_k}(\alpha^+(n_k)) - H_{n_k}(\alpha^-(n_k))) = \sum_k \epsilon(k)(H_{n_k}(x_k) - H_{n_k}(0)) \text{ converges.}$$

Comparing with $\lim_{k\to\infty} \|H_{n_k}(x_k) - H_{n_k}(0)\|_Y = 0$, we have $(\psi(\alpha^+) - \psi(\alpha^-)) \in \mathrm{cs}(Y)$. A contradiction! $\square$

Theorem 5.9. *For $m \geq 1$, we have*

(i) $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m+1)}$,
(ii) $\mathbb{R}^{\mathbb{N}}/c_0 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

Proof. Clause (i) is trivial. Already known $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}$, Theorem 4.4 gives $\mathbb{R}^{\mathbb{N}}/c_0 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}$. For proving $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$, we only need to prove

$$(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) <_B (c_0)^{\mathbb{N}}/\mathrm{cs}(c_0).$$

Since $(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \leq_B (c_0)^{\mathbb{N}}/\mathrm{cs}(c_0)$ is trivial, the assertion follows from Lemma 5.8. $\square$

5.2. **Bases of ℓ_1.** Similar to last subsection, we know $\mathbb{R}^{\mathbb{N}}/\ell_1$ is the minimum element among $E(\ell_1, (x_n))$ with (x_n) a basis of ℓ_1. Unfortunately, we did not find a maximum for them so far. We managed to find some bases such that the equivalence relations generated by them are not Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_1$.

We denote $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$ where

$$\mathrm{bv} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |a(n) - a(n+1)| < +\infty\}.$$

Let $y_0^1 = e_0$ and $y_n^1 = e_n - e_{n-1}$ for $n > 0$, where (e_n) is the unit vector basis of ℓ_1. Then (y_n^1) is a basis of ℓ_1 too. Since

$$\sum_n a(n)y_n^1 = (a(0) - a(1), a(1) - a(2), \cdots, a(n) - a(n+1), \cdots),$$

we can see $\mathrm{bv}_0 = \mathrm{coef}(\ell_1, (y_n^1))$.

Theorem 5.10. $\mathbb{R}^{\mathbb{N}}/\ell_1 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$.

Proof. For $a \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$, define $\theta(a)(2k) = a(k)$ and $\theta(a)(2k+1) = 0$. It is clear that θ is a Borel reduction of $\mathbb{R}^{\mathbb{N}}/\ell_1$ to $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$.

Note that $\|y_0^1 + \cdots + y_n^1\| = 1$ for any $n \in \mathbb{N}$, but $\sum_n y_n^1$ does not converge. It follows that (y_n^1) is not boundedly complete. By Corollary 3.3, we have $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$. And hence $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_1$. □

For $m \geq 1$, similar to $\mathrm{cs}^{(m)}$, note that $\bigoplus_{i=1}^m \ell_1 \cong \ell_1$, we choose a suitable basis (y_n^m) of ℓ_1 such that $\mathrm{coef}(\ell_1, (y_n^m)) = \mathrm{bv}_0^{(m)} = \mathrm{bv}^{(m)} \cap c_0$, where

$$\mathrm{bv}^{(m)} = \{a \in \mathbb{R}^{\mathbb{N}} : \forall i \leq m (\sum_j |a(mj+i-1) - a(m(j+1)+i-1)| < +\infty)\}.$$

Also similar to $\mathrm{cs}^{(\infty)}$, we define $\mathrm{bv}_0^{(\infty)}$ as follows. Recall that

$$\left(\bigoplus_{i\in\mathbb{N}} \ell_1\right)_1 = \{(a_n) \in (\ell_1)^{\mathbb{N}} : \sum_n \|a_n\| < +\infty\}.$$

We still have $\left(\bigoplus_{i\in\mathbb{N}} \ell_1\right)_1 \cong \ell_1$. Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$ such that, for any i, $\langle i, j\rangle$ is strictly increasing with respect to variable j. We can find a basis (y_n^∞) of ℓ_1 such that $\mathrm{coef}(\ell_1, (y_n^\infty)) = \mathrm{bv}_0^{(\infty)} = \mathrm{bv}^{(\infty)} \cap c_0$, where

$$\mathrm{bv}^{(\infty)} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_i \sum_j |a(\langle i, j\rangle) - a(\langle i, j+1\rangle)| < +\infty\}.$$

It is trivial that

$$\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(m)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(m+1)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)}.$$

We do not know whether they are Borel bireducible with each other. We also compare them with the equivalence relations appear in last subsection.

Theorem 5.11. *For any $m \in \mathbb{N}$, we have*

(i) $\mathbb{R}^{\mathbb{N}}/\ell_1 \not\leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$,
(ii) $\mathbb{R}^{\mathbb{N}}/c_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)}$,
(iii) $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)} <_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$.

Proof. (i) The proof is combined proofs of Theorem 4.4 and Lemma 5.8, So we omit some similar arguments.

Since $(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$, we assume for contradiction that $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B (\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m)$. Denote $F_n = \{i/2^n : i = 0, 1, \cdots, 2^n\}$. From Lemma 4.2, we can find an infinite set $I \subseteq \mathbb{N}$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to (\mathbb{R}^m)^{l_n}$ for each $n \in I$, satisfying the following requirements. Letting by (n_k) the strictly increasing enumeration of I, we define ψ as

$$\psi(a) = H_{n_0}(a(n_0))^\frown H_{n_1}(a(n_1))^\frown H_{n_2}(a(n_2))^\frown \cdots,$$

for any $a \in \prod_{n\in I} F_n$, then we have, for $a, b \in \prod_{n\in I} F_n$,

$$\sum_{n\in I} |a(n) - b(n)| < +\infty \iff (\psi(a) - \psi(b)) \in \mathrm{cs}(\mathbb{R}^m).$$

Choose an $i_n \in \{0, 1, \cdots, 2^n\}$ for each $n \in I$ such that

$$\lim_{k\to\infty} \frac{i_{n_k}}{2^{n_k}} = 0, \quad \sum_{n\in I} \frac{i_n}{2^n} = +\infty.$$

Now denote $x_k = i_{n_k}/2^{n_k}$ for each $k \in \mathbb{N}$. The rest part of proof is almost word for word a copy of the proof of Lemma 5.8.

(ii) By Theorem 8.5.2 and Lemma 8.5.3 of [7], we only need to prove that $\mathrm{bv}_0^{(\infty)}$ is $\mathbf{\Sigma}_3^0$. Define a subset $A \subseteq \mathbb{R}^{\mathbb{N}}$ satisfying that, for any $a \in \mathbb{R}^{\mathbb{N}}$,

$$\begin{aligned} a \in A \quad &\iff \forall i \in \mathbb{N}(\text{there is a subsequence of } a(\langle i, \cdot\rangle) \text{ converges to } 0) \\ &\iff \forall i, p, N \in \mathbb{N} \exists j > N(|a(\langle i, j\rangle)| < 1/p). \end{aligned}$$

Note that, for any $a \in \mathrm{bv}^{(\infty)}$, we have $\lim_{j\to\infty} a(\langle i, j\rangle)$ converges for each $i \in \mathbb{N}$, and $a(\langle i, \cdot\rangle)$ is uniformly convergent to 0 as $i \to \infty$. It follows that $\mathrm{bv}_0^{(\infty)} = \mathrm{bv}^{(\infty)} \cap c_0 = \mathrm{bv}^{(\infty)} \cap A$. Since $\mathrm{bv}^{(\infty)}$ is F_σ and A is G_δ, we see that $\mathrm{bv}_0^{(\infty)}$ is $\mathbf{\Delta}_3^0$. Indeed, $\mathrm{bv}_0^{(\infty)}$ is a $D_2(\mathbf{\Sigma}_2^0)$ set (for definition of $D_2(\mathbf{\Sigma}_2^0)$, see, e.g., [15], 22.E).

(iii) $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)} <_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$ follows from Theorem 5.3 and (ii). $\square$

However, we do not know whether $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

For $p > 1$, (y_n^1) is also a sequence in ℓ_p, but not a basis of ℓ_p. We have

$$\mathrm{coef}(\ell_p, (y_n^1)) = c_0 \cap \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |a(n) - a(n+1)|^p < +\infty\}.$$

For the Borel reducibility, we claim $E(\ell_p, (y_n^1)) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_p \otimes \mathbb{R}^{\mathbb{N}}/c_0$. This is because of Theorem 5.3 and the following θ which witnesses $\mathbb{R}^{\mathbb{N}}/\ell_p \otimes [0,1]^{\mathbb{N}}/c_0 \leq_B E(\ell_p, (y_n^1))$. For $a \in \mathbb{R}^{\mathbb{N}}, b \in [0,1]^{\mathbb{N}}$ and $n \in \mathbb{N}$, we define

$$\theta(a,b)(n) = \begin{cases} b(0), & 0 \leq n \leq 3, \\ a(k) + b(k), & n = 2^{k+2}, \\ b(k) + \frac{(i-1)(b(k+1)-b(k))}{2^{k+2}-2}, & n = 2^{k+2} + i, 1 \leq i < 2^{k+2}. \end{cases}$$

Comparing with Theorem 4.4, we are interested in these examples because the unit vector basis of $\mathrm{coef}(\ell_p, (y_n^1))$ is conditional while $\mathbb{R}^{\mathbb{N}}/\ell_p \otimes \mathbb{R}^{\mathbb{N}}/c_0$ is generated by an unconditional basis.

5.3. **Rearrangements of bases.**

Lemma 5.12. *Let (x_n) be a basis of a Banach space X, π a permutation on $\mathbb{N}$. If $(x_{\pi(n)})$ is also a basis, then $E(X, (x_n)) \sim_B E(X, (x_{\pi(n)}))$.*

Proof. We define $\theta(a)(n) = a(\pi(n))$ for $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $a, b \in \mathbb{R}^{\mathbb{N}}$. If $a - b \in \mathrm{coef}(X, (x_n))$, we denote $x = \sum_n (a(n) - b(n))x_n$. Since $(x_{\pi(n)})$ is also a basis, we have

$$x = \sum_n x^*_{\pi(n)}(x) x_{\pi(n)} = \sum_n (a(\pi(n)) - b(\pi(n))) x_{\pi(n)}.$$

So $\theta(a) - \theta(b) \in \mathrm{coef}(X, (x_{\pi(n)}))$. By the same arguments, we can show that $\theta(a) - \theta(b) \in \mathrm{coef}(X, (x_{\pi(n)}))$ implies $a - b \in \mathrm{coef}(X, (x_n))$ too. $\square$

Corollary 5.13. *Let (x_n) be an unconditional basis of a Banach space X, then for any permutation π on $\mathbb{N}$, we have $E(X,(x_n)) \sim_B E(X,(x_{\pi(n)}))$.*

Now we consider rearrangements of the bases (x_n^m) and (x_n^∞) of c_0. Since they are conditional bases, there must be some rearrangements are not bases. However, we always have:

Theorem 5.14. *Let $\pi:\mathbb{N}\to\mathbb{N}$ be a permutation. For $m\geq 1$ or $m=\infty$, we have $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} \sim_B E(c_0,(x^m_{\pi(n)}))$.*

Proof. By Theorem 5.3, we only need to show $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} \otimes \mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$. We only prove for $m=1$, since other cases are similar. For any $a,b\in\mathbb{R}^{\mathbb{N}}$ and $n\in\mathbb{N}$, define

$$\theta(a,b)(n)=\begin{cases} a(k), & n=3k,\\ b(k), & n=3k+1,\\ -b(k), & n=3k+2.\end{cases}$$

It is easy to see that θ is a disired Borel reduction. $\square$

The situation is different when we consider rearrangements of the basis (y_n^1) of ℓ_1. We only present a special rearrangement of (y_n^1) as:

$$y_0^1,y_2^1,y_1^1,\quad y_4^1,y_6^1,y_3^1,\quad y_8^1,y_{10}^1,y_5^1,\quad \cdots.$$

More precisely, define a permutation $\pi_0:\mathbb{N}\to\mathbb{N}$ as follows:

$$\pi_0(n)=\begin{cases} 4k, & n=3k,\\ 4k+2, & n=3k+1,\\ 2k+1, & n=3k+2,\end{cases}$$

then we consider the rearranged sequence $(y^1_{\pi_0(n)})$.

Example 5.15. $E(\ell_1,(y^1_{\pi_0(n)})) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$, i.e., $E(\ell_1,(y^1_{\pi_0(n)}))$ has the highest allowable complexity of Theorem 5.3.

Proof. It is well known that $\mathbb{R}^{\mathbb{N}}/c_0 \sim_B [0,1]^{\mathbb{N}}/c_0$. Thus, by Theorem 5.3, we only need to show $\mathbb{R}^{\mathbb{N}}/\ell_1 \otimes [0,1]^{\mathbb{N}}/c_0 \leq_B E(\ell_1,(y^1_{\pi_0(n)}))$. For any $a\in\mathbb{R}^{\mathbb{N}}, b\in [0,1]^{\mathbb{N}}$, and $j\in\mathbb{N}$, we define

$$\theta(a,b)(j)=\begin{cases} a(l), & \pi_0(j)=2^{l+1},\\ (2^{l+1}-1)^{-1}, & 2^{l+2}+2\leq \pi_0(j)\leq 2^{l+2}+2[(2^{l+1}-1)b(l)],\\ 0, & \text{otherwise.}\end{cases}$$

Let $a_1,a_2\in\mathbb{R}^{\mathbb{N}}$ and $b_1,b_2\in[0,1]^{\mathbb{N}}$. Denote $d=\theta(a_1,b_1)-\theta(a_2,b_2)$. Then

$$d\in \mathrm{coef}(\ell_1,(y^1_{\pi_0(n)})) \iff \begin{cases} (d(\pi_0^{-1}(n)))\in \mathrm{bv}_0=\mathrm{bv}\cap c_0,\\ \sum_j d(j)y^1_{\pi_0(j)}=\sum_n d(\pi_0^{-1}(n))y^1_n.\end{cases}$$

It is easy to see that

$$(d((\pi_0^{-1}(n)))\in c_0 \iff d\in c_0 \iff a_1-a_2\in c_0,$$

$$\begin{aligned}(d(\pi_0^{-1}(n)))\in \mathrm{bv} &\iff 2\sum_l \left(|a_1(l)-a_2(l)|+(2^{l+1}-1)^{-1}\right)<+\infty\\ &\iff a_1-a_2\in\ell_1.\end{aligned}$$

Thus

$$(d(\pi_0^{-1}(n)))\in \mathrm{bv}_0 \iff a_1-a_2\in\ell_1.$$

Now suppose $a_1-a_2\in\ell_1$. Then there exists $x\in\ell_1$ such that $x=\sum_n d(\pi_0^{-1}(n))y_n^1$. Note that $\lim_{j\to\infty} d(j)=0$. We have

$$\sum_j d(j)y^1_{\pi_0(j)}=x \iff \lim_{k\to\infty}\sum_{j<3k} d(j)y^1_{\pi_0(j)}=x.$$

By the definition of π_0,

$$\sum_{j<3k} d(j)y^1_{\pi_0(j)}=\sum_{n<2k} d(\pi_0^{-1}(n))y_n^1+\sum_{k\le i<2k} d(\pi_0^{-1}(2i))y^1_{2i}.$$

If $k=2^{l+1}$ for some $l\in\mathbb{N}$, then

$$\begin{aligned}&\|\textstyle\sum_{2^{l+1}\le i<2^{l+2}} d(\pi_0^{-1}(2i))y^1_{2i}\|\\ =\;& 2|a_1(l+1)-a_2(n+1)|+2(2^{l+1}-1)^{-1}\left|[(2^{l+1}-1)b_1(l)]-[(2^{l+1}-1)b_2(l)]\right|;\end{aligned}$$

otherwise, we have $2^{l+1}<k<2^{l+2}$ for some $l\in\mathbb{N}$, then

$$\begin{aligned}&\|\textstyle\sum_{k\le i<2k} d(\pi_0^{-1}(2i))y^1_{2i}\|\\ \le\;& \|\textstyle\sum_{k\le i<2^{l+2}} d(\pi_0^{-1}(2i))y^1_{2i}\|+\|\sum_{2^{l+2}\le i<2k} d(\pi_0^{-1}(2i))y^1_{2i}\|\\ \le\;& \|\textstyle\sum_{2^{l+1}\le i<2^{l+2}} d(\pi_0^{-1}(2i))y^1_{2i}\|+\|\sum_{2^{l+2}\le i<2^{l+3}} d(\pi_0^{-1}(2i))y^1_{2i}\|.\end{aligned}$$

Hence

$$\sum_j d(j)y^1_{\pi_0(j)}=x \iff \lim_{k\to\infty}\sum_{k\le i<2k} d(\pi_0^{-1}(2i))y^1_{2i}=0 \iff b_1-b_2\in c_0.$$

To sum up, θ is a desired Borel reduction. □

6. James' space and F.D.D. equivalence relations

James' space J serves as an example of a non-reflexive space whose double dual is isomorphic to itself. Furthermore, it is also an example of a space with a basis has no unconditional basis (see, e.g., [16], Example 1.d.2). In this section, we wish to compare $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/J$. For this purpose, F.D.D. equivalence relations turn out to be an useful tool.

For $a\in\mathbb{R}^{\mathbb{N}}$, denote

$$\begin{aligned}\|a\|_J=\frac{1}{\sqrt{2}}\sup[(a(n_1)-a(n_2))^2+(a(n_2)-a(n_3))^2+\cdots\\ +(a(n_{m-1})-a(n_m))^2+(a(n_m)-a(n_1))^2]^{1/2},\end{aligned}$$

where the supremum taken over all choices of m and $n_1<n_2<\cdots<n_m$. Then James' space defined as

$$J=c_0\cap\{a\in\mathbb{R}^{\mathbb{N}}:\|a\|_J<+\infty\}.$$

Since the unit vector basis of J is not boundedly complete, by Corollary 3.3, J is not F_σ. Similar to (ii) of proof of Theorem 5.11, we can see that J is $\mathbf{\Delta}^0_3$. These imply that

$$\mathbb{R}^{\mathbb{N}}/J \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty, \quad \mathbb{R}^{\mathbb{N}}/c_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/J.$$

Recall that ℓ_p^n is $\mathbb{R}^n$ equipped with the ℓ_p norm. By the same sprit, we denote by J^n the n-dimensional space equipped with the following norm:

$$\begin{array}{rl} & \|(r_0,\cdots,r_{n-1})\|_J \stackrel{\text{Def}}{=} \|(r_0,\cdots,r_{n-1},0,0,\cdots)\|_J \\ = & \frac{1}{\sqrt{2}}\sup[(r_{n_1}-r_{n_2})^2+\cdots+(r_{n_{m-1}}-r_{n_m})^2+r_{n_m}^2+r_{n_1}^2]^{1/2}, \end{array}$$

where the supremum taken over all choices of m and $n_1<n_2<\cdots<n_m<n$. Note that (J^n) is a finite dimensional decomposition of $(\bigoplus_{n\geq 1} J^n)_2$. Now we consider the F.D.D. equivalence relation $E((\bigoplus_{n\geq 1} J^n)_2,(J^n))$.

Let $(e_{n_k^*})$ be a subsequence of the unit vector basis (e_n) such that, if $k=1+2+\cdots+l$ for some $l\geq 1$, then $1+n_{k-1}^*<n_k^*$; otherwise $1+n_{k-1}^*=n_k^*$. The following is one of such examples:

$$e_0, \quad e_2,e_3, \quad e_5,e_6,e_7, \quad e_9,e_{10},e_{11},e_{12}, \quad \cdots.$$

It is straightforward to check that there is a canonical Lipschitz isomorphic Φ from $(\bigoplus_{n\geq 1} J^n)_2$ to the closed linear span $[e_{n_k^*}]_{k\in\mathbb{N}}$ satisfying $\|x\|\leq\|\Phi(x)\|_J\leq\sqrt{2}\|x\|$ for $x\in(\bigoplus_{n\geq 1} J^n)_2$. Thus

$$E(\left(\bigoplus_{n\geq 1} J^n\right)_2,(J^n))\sim_B E(J,(e_{n_k^*})).$$

Recall that a basis (x_n) of a Banach space X is *symmetric* if, for any permutation $\pi:\mathbb{N}\to\mathbb{N}$, $(x_{\pi(n)})$ is equivalent to (x_n). It is well known that every symmetric basis is unconditional and semi-normalized. From Proposition 3.a.3 of [16], we know that, for any injection $\sigma:\mathbb{N}\to\mathbb{N}$, $(x_{\sigma(n)})$ is also equivalent to (x_n).

Lemma 6.1. *Let (x_n) be a symmetric basis of Banach space X. Then*

$$E(X,(x_n))\sim_B [0,1]^{\mathbb{N}}/\mathrm{coef}(X,(x_n)).$$

Proof. Fix a bijection $\langle\cdot,\cdot\rangle:\mathbb{N}\times\mathbb{Z}\to\mathbb{N}$. For any $a\in\mathbb{R}^{\mathbb{N}}$ and $k,l\in\mathbb{N}$, define

$$\theta(a)(\langle k,l\rangle)=\begin{cases} 0, & a(k)<l, \\ a(k)-l, & l\leq a(k)<l+1, \\ 1, & a(k)\geq l+1. \end{cases}$$

For any $a,b\in\mathbb{R}^{\mathbb{N}}$, we split $\mathbb{N}$ into three sets

$$\begin{aligned} I_0 &= \{k\in\mathbb{N}:[a(k)]=[b(k)]\}, \\ I_1 &= \{k\in\mathbb{N}:|[a(k)]-[b(k)]|=1\}, \\ I_2 &= \{k\in\mathbb{N}:|[a(k)]-[b(k)]|\geq 2\}. \end{aligned}$$

For $k\in I_0$, denote $l_k=[a(k)]=[b(k)]$. Then

$$(\theta(a)-\theta(b))(\langle k,l_k\rangle)=a(k)-b(k),$$

while $(\theta(a)-\theta(b))(\langle k,l\rangle)=0$ for $l\neq l_k$.

For $k\in I_1$, denote $l_k=\max\{[a(k)],[b(k)]\}$. Then

$$|(\theta(a)-\theta(b))(\langle k,l_k\rangle)|+|(\theta(a)-\theta(b))(\langle k,l_k-1\rangle)|=|a(k)-b(k)|,$$

while $(\theta(a)-\theta(b))(\langle k,l\rangle)=0$ for $l\neq l_k,l_k-1$.

For $k\in I_2$, we have $|a(k)-b(k)|\geq 1$ and $|(\theta(a)-\theta(b))(\langle k,l\rangle)|=1$ for some l.

If I_2 is infinite, since (x_n) is semi-normalized, neither $a-b$ nor $\theta(a)-\theta(b)$ is in $\mathrm{coef}(X,(x_n))$. If I_2 is finite, without loss of generality, we can assume $I_2=\emptyset$, and both I_0, I_1 are infinite. Because (x_n) is unconditional, by Proposition 1.c.6 of [16], we have

$$\begin{array}{ll} & \sum_k(a(k)-b(k))x_k \text{ converges}\\ \iff & \sum_k|a(k)-b(k)|x_k \text{ converges}\\ \iff & \sum_{k\in I_0}|a(k)-b(k)|x_k,\sum_{k\in I_1}|a(k)-b(k)|x_k \text{ are convergent.}\end{array}$$

Still because (x_n) is unconditional, the convergency of $\sum_{k\in I_1}|a(k)-b(k)|x_k$ is equivalent to both

$$\sum_{k\in I_1}|(\theta(a)-\theta(b))(\langle k,l_k\rangle)|x_k,\quad \sum_{k\in I_1}|(\theta(a)-\theta(b))(\langle k,l_k-1\rangle)|x_k$$

are convergent. Their convergency are equivalent to both

$$\sum_{k\in I_1}|(\theta(a)-\theta(b))(\langle k,l_k\rangle)|x_{\langle k,l_k\rangle},\quad \sum_{k\in I_1}|(\theta(a)-\theta(b))(\langle k,l_k-1\rangle)|x_{\langle k,l_k-1\rangle}$$

are convergent, since (x_n) is symmetric and both $k\mapsto\langle k,l_k\rangle$, $k\mapsto\langle k,l_k-1\rangle$ are injection. By the same reason

$$\sum_{k\in I_0}|a(k)-b(k)|x_k \text{ converges }\iff\sum_{k\in I_0}|(\theta(a)-\theta(b))(\langle k,l_k\rangle)|x_{\langle k,l_k\rangle}\text{ converges.}$$

Since $(\theta(a)-\theta(b))(n)=0$ for any n outside the range of $\langle k,l_k\rangle$ and $\langle k,l_k-1\rangle$, we get

$$\begin{array}{ll} & \sum_k(a(k)-b(k))x_k \text{ converges}\\ \iff & \sum_n|(\theta(a)-\theta(b))(n)|x_n \text{ converges}\\ \iff & \sum_n((\theta(a)-\theta(b))(n))x_n \text{ converges}\end{array}$$

as desired. □

Theorem 6.2. *Let (x_n) be a symmetric basis of Banach space X. Then*

$$E(X,(x_n))\leq_B\mathbb{R}^{\mathbb{N}}/J\iff E(X,(x_n))\leq_B E(\left(\bigoplus_{n\geq 1}J^n\right)_2,(J^n)).$$

Proof. "$\Leftarrow$" follows from $E(J,(e_{n_k^*}))\leq_B\mathbb{R}^{\mathbb{N}}/J$. We only prove the "$\Rightarrow$" side.

Denote $F_n=\{i/2^n:i=0,1,\cdots,2^n\}$. Since $E(X,(x_n))\leq_B\mathbb{R}^{\mathbb{N}}/J$, from Lemma 4.2, we can find an infinite set $I\subseteq\mathbb{N}$, a natural number $l_n\geq 1$ and

a map $H_n : F_n \to \mathbb{R}^{l_n}$ for each $n \in I$, satisfying the following requirements. Letting by (n_k) the strictly increasing enumeration of I, we define ψ as

$$\psi(a) = H_{n_0}(a(n_0))^\frown H_{n_1}(a(n_1))^\frown H_{n_2}(a(n_2))^\frown \cdots,$$

for any $a \in \prod_{n\in I} F_n$, then we have, for $a, b \in \prod_{n\in I} F_n$,

$$\sum_{n\in I}(a(n) - b(n))x_n \text{ converges} \iff (\psi(a) - \psi(b)) \in J.$$

Now we define, for any $a \in \prod_k F_{n_{2k}}$,

$$\psi'(a) = H_{n_0}(a(0))^\frown(\overbrace{0, \cdots, 0}^{l_{n_1}})^\frown H_{n_2}(a(n_2))^\frown(\overbrace{0, \cdots, 0}^{l_{n_3}})^\frown \cdots.$$

Set $s_k = l_{n_0} + l_{n_1} + \cdots + l_{n_k}$. Let $(e_{n'_k})$ be the following subsequence of (e_n):

$$e_0, e_1 \cdots, e_{s_0-1}, \quad e_{s_2}, e_{s_2+1}, \cdots, e_{s_3-1}, \quad e_{s_4}, e_{s_4+1}, \cdots, e_{s_5-1}, \quad \cdots.$$

We still have, for $a, b \in \prod_k F_{n_{2k}}$,

$$\sum_k (a(k) - b(k))x_{n_{2k}} \text{ converges} \iff (\psi'(a) - \psi'(b)) \in [e_{n'_k}]_{k\in\mathbb{N}}.$$

Therefore, ψ' is a Borel reduction of $\prod_k F_{n_{2k}}$ to $E(J, (e_{n'_k}))$. Note that $[e_{n'_k}]$ is Lipschtz isomorphic to $\left(\bigoplus_k J^{l_{2k}}\right)_2$, thus Lipschitz embeds into $(\bigoplus_{n\geq1} J^n)_2$. We get

$$\prod_k F_{n_{2k}}/\mathrm{coef}(X, (x_{n_{2k}})) \leq_B E(J, (e_{n'_k})) \leq_B E(\left(\bigoplus_{n\geq1} J^n\right)_2, (J^n)).$$

Since (x_n) is symmetric, $\mathrm{coef}(X, (x_{n_{2k}})) = \mathrm{coef}(X, (x_k))$. Furthermore, since (x_k) is semi-normalized and $\sum_k 1/2^{n_{2k}} < +\infty$, we have

$$[0, 1]^{\mathbb{N}}/\mathrm{coef}(X, (x_k)) \sim_B \prod_k F_{n_{2k}}/\mathrm{coef}(X, (x_k)).$$

Then Lemma 6.1 gives the required result. $\square$

Theorem 6.3. *For $p \geq 1$, we have $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/J \iff p \leq 2$.*

Proof. Because ℓ_p is symmetric, by Theorem 6.2, we only need to consider $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E((\bigoplus_{n\geq1} J^n)_2, (J^n))$. Since $\mathbb{R}^{\mathbb{N}}/\ell_2 \leq_B E((\bigoplus_{n\geq1} J^n)_2, (J^n))$ is trivial, and by Dougherty-Hjorth's theorem, $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_2$ for any $p \leq 2$, we finish the "$\Leftarrow$" side.

For proving the "$\Rightarrow$" side, suppose $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E((\bigoplus_{n\geq1} J^n)_2, (J^n))$. In Definition 3.2 of [3], the equivalence relation $E((\bigoplus_{n\geq1} J^n)_2, (J^n))$ was denoted as $E((J^n)_{n\in\mathbb{N}}; 2)$. It is clear that $E((J^n)_{n\in\mathbb{N}}; 2) \leq_B E(J; 2)$, so we have $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E(J; 2)$. From Theorem 4.8 and Lemma 5.1 of [3], we get $p \leq 2$. $\square$

For proving the following theorem, we need a notion of ultraproduct of Banach space. An ultrafilter $\mathfrak{A}$ on $\mathbb{N}$ is called free if it does not contain any finite set. Let X be a Banach space. Consider the space $\ell_\infty(X)$ of all bounded sequences $\alpha \in X^{\mathbb{N}}$ with the norm $\|\alpha\| = \sup_n \|\alpha(n)\|$. Its subspace $N = \{\alpha : \lim_{\mathfrak{A}} \|\alpha(n)\| = 0\}$ is closed. The ultraproduct $(X)_{\mathfrak{A}}$ is the quotient space $\ell_\infty(X)/N$ with the norm $\|(\alpha)_{\mathfrak{A}}\|_{\mathfrak{A}} = \lim_{\mathfrak{A}} \|\alpha(n)\|$. For more details on ultraproducts in Banach space theory, see [10].

Theorem 6.4. $\mathbb{R}^{\mathbb{N}}/\ell_2 <_B E((\bigoplus_{n\geq 1} J^n)_2, (J^n)) <_B \mathbb{R}^{\mathbb{N}}/J$.

Proof. We only need to prove $E((\bigoplus_{n\geq 1} J^n)_2, (J^n)) \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_2$ and $\mathbb{R}^{\mathbb{N}}/J \not\leq_B E((\bigoplus_{n\geq 1} J^n)_2, (J^n))$. The second one is because that, by Theorem 3.2, the equivalence relation $E((\bigoplus_{n\geq 1} J^n)_2, (J^n))$ is Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$, while $\mathbb{R}^{\mathbb{N}}/J$ is not.

Assume for contradiction that $E((\bigoplus_{n\geq 1} J^n)_2, (J^n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_2$. By Theorem 4.8 of [3], J is finitely Lipschitz embeds (see Definition 4.7 of [3]) into $\ell_2(\ell_2) \cong \ell_2$. Fix a sequence of finite subsets (F_n) of J such that

$$\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

and $\bigcup_n F_n$ is dense in J. By the finitely Lipschitz embeddability, there exist $A > 0$ and $T_n : F_n \to \ell_2$ satisfying, for $a, b \in F_n$,

$$A^{-1}\|a - b\|_J \leq \|T_n(a) - T_n(b)\| \leq A\|a - b\|_J.$$

Without loss of generality, we many assume that $T_n(0) = 0$ for each n. Fix a free ultrafilter $\mathfrak{A}$ on $\mathbb{N}$. For any $a \in \bigcup_n F_n$, set $m = \min\{n : a \in F_n\}$. Since $\|T_n(a)\| \leq A\|a\|_J$ for $n \geq m$, we can define

$$T(a) = (\overbrace{0, \cdots, 0}^{m}, T_m(a), T_{m+1}(a), \cdots, T_n(a), \cdots)_{\mathfrak{A}}.$$

By the definition of the norm on $(\ell_2)_{\mathfrak{A}}$, it is easy to see that, for any $a, b \in \bigcup_n F_n$,

$$A^{-1}\|a - b\|_J \leq \|T(a) - T(b)\|_{\mathfrak{A}} \leq A\|a - b\|_J.$$

Since $\bigcup_n F_n$ is dense in J, T can be extended to a Lipschitz embedding $\tilde{T} : J \to (\ell_2)_{\mathfrak{A}}$. Note that $(\ell_2)_{\mathfrak{A}}$ is still a Hilbert space (actually nonseparable, see, e.g., Proposition F.3 of [1]), so it is reflexive. By Corollary 7.10 of [1], J is isomorphic to a closed subspace of $(\ell_2)_{\mathfrak{A}}$. This is impossible, because J is not reflexive. □

A well known generalization of James' space is v_p^0 for $p > 1$ (cf. [20]). For $a \in \mathbb{R}^{\mathbb{N}}$, denote

$$\begin{aligned}\|a\| = 2^{-1/p} \sup[&|a(n_1) - a(n_2)|^p + |a(n_2) - a(n_3)|^p + \cdots \\ &+ |a(n_{m-1}) - a(n_m)|^p + |a(n_m) - a(n_1)|^p]^{1/p},\end{aligned}$$

where the supremum taken over all choices of m and $n_1 < n_2 < \cdots < n_m$. Define $v_p = \{a \in \mathbb{R}^{\mathbb{N}} : \|a\| < +\infty\}$ and $v_p^0 = c_0 \cap v_p$. Then $J = v_2^0$.

Similar proof gives

(i) Let (x_n) be a symmetric basis of Banach space X. Then

$$E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/v_p^0 \iff E(X,(x_n)) \leq_B E(\left(\bigoplus_{n\geq 1} v_p^n\right)_p,(v_p^n)).$$

(ii) For $q \geq 1$, we have $\mathbb{R}^{\mathbb{N}}/\ell_q \leq_B \mathbb{R}^{\mathbb{N}}/v_p^0 \iff q \leq p$.

(iii) $\mathbb{R}^{\mathbb{N}}/\ell_p <_B E((\bigoplus_{n\geq 1} v_p^n)_p,(J^n)) <_B \mathbb{R}^{\mathbb{N}}/v_p^0$.

Furthermore, if we extend the definition of v_p^0 to $p=1$, the resulted space is just the space bv_0. For symmetric basis (x_n) of X, we still have

$$E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \iff E(X,(x_n)) \leq_B E(\left(\bigoplus_{n\geq 1} \mathrm{bv}_0^n\right)_1,(\mathrm{bv}_0^n)).$$

Unlike the previous clause (iii), we have $E((\bigoplus_{n\geq 1}\mathrm{bv}_0^n)_1,(\mathrm{bv}_0^n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_1$. A desired Borel reduction θ defined as, for $\alpha \in \prod_{n\geq 1}\mathrm{bv}_0^n$,

$$\theta(\alpha) = (\alpha(0),\alpha(1)_0-\alpha(1)_1,\alpha(1)_1,\alpha(2)_0-\alpha(2)_1,\alpha(2)_1-\alpha(2)_2,\alpha(2)_2,\cdots).$$

Therefore, though $\mathbb{R}^{\mathbb{N}}/\ell_1 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$, we still have, for symmetric basis (x_n) of X,

$$E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \iff E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_1.$$

7. Further remarks

Perhaps the most interesting question is:

Question 7.1. If (x_n) and (y_n) are two bases of a reflexive Banach space X, does $E(X,(x_n)) \sim_B E(Y,(y_n))$?

Let us denote by SER the set of all Schauder equivalence relations. Ma [17] showed that, in SER, there is a maximum element and $\mathbb{R}^{\mathbb{N}}/c_0$ is a minimal element with respect to Borel reducibility. Hjorth's dichotomy below $\mathbb{R}^{\mathbb{N}}/\ell_1$ (see Corollary 5.6 of [11]) implies that $\mathbb{R}^{\mathbb{N}}/\ell_1$ is another minimal element in SER. If we restrict attention on two main classes of spaces X whose unit vector basis is symmetric, i.e., Orlicz sequence spaces and Lorentz sequence spaces, we always have either $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B \mathbb{R}^{\mathbb{N}}/X$ or $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/X$. Therefore, our first question is:

Question 7.2. Let (x_n) be a symmetric basis of X, does either $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B E(X,(x_n))$ or $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(X,(x_n))$ hold?

We say two elements $E,F \in$ SER are incompatible in SER, if no element in SER can be Borel reducible to both E and F. It is well known that $\mathbb{R}^{\mathbb{N}}/\ell_1$ and $\mathbb{R}^{\mathbb{N}}/c_0$ form an incompatible pair. Ma [17] also indicated, Farah [5] potentially proved that, for any α-Tsirelson space T_α, $\mathbb{R}^{\mathbb{N}}/T_\alpha$ are incompatible with either $\mathbb{R}^{\mathbb{N}}/\ell_1$ or $\mathbb{R}^{\mathbb{N}}/c_0$, furthermore, whenever $\alpha \neq \beta$, we have $\mathbb{R}^{\mathbb{N}}/T_\alpha$ and $\mathbb{R}^{\mathbb{N}}/T_\beta$ are incompatible.

Let (x_n) be an unconditional basis of a Banach space X. James [12] proved that, if (x_n) is not boundedly complete, then there exists a block basis (u_k) of

(x_n) such that $\mathrm{coef}(X,(u_k)) = c_0$ (see also [16], Theorem 1.c.10). Comparing with Corollary 3.3, we get a dichotomy that, for unconditional basis (x_n) of X, exactly one of the following holds:

(i) $E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$,
(ii) $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(X,(x_n))$.

This dichotomy cannot be generalized to conditional basis, since either $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$ or $\mathbb{R}^{\mathbb{N}}/J$ can serve as a counterexample. This is because clause (i) is equivalent to say that $\mathrm{coef}(X,(x_n))$ is F_σ, while clause (ii) implies it is $\mathbf{\Pi}^0_3$-complete, and bv_0 and J are both $D_2(\mathbf{\Sigma}^0_2)$. Thus we ask two related questions:

Question 7.3. (I) Is there a basis (x_n) of X such that $\mathrm{coef}(X,(x_n))$ is $\mathbf{\Delta}^0_3$ but not $D_2(\mathbf{\Sigma}^0_2)$?
(II) For any basis (x_n) of X, if $\mathrm{coef}(X,(x_n))$ is a $\mathbf{\Pi}^0_3$-complete set in $\mathbb{R}^{\mathbb{N}}$, does $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(X,(x_n))$?

We can use an unconditional basis (x_n) of X to generate an ideal on $\mathbb{N}$. Denote $\mathcal{I}(X,(x_n)) = \{A \subseteq \mathbb{N} : \sum_{n\in A} x_n \text{ converges}\}$. It is clear that $P(\mathbb{N})/\mathcal{I}(X,(x_n)) \leq_B E(X,(x_n))$. Let (x_n) and (y_n) be unconditional bases of X and Y respectively, and suppose $E(X,(x_n)) \leq_B E(Y,(y_n))$. Applying Lemma 4.2 on $\{0,1\}^{\mathbb{N}}$, we can find a subsequence (x_{n_k}) of (x_n) and a block basis (u_k) of (y_n) such that $\mathcal{I}(X,(x_{n_k})) = \mathcal{I}(Y,(u_k))$. Furthermore, for any block basis (v_k) of (x_n), we can find a subsequence of (v_k) and a block basis of (y_n) such that they generate the same ideal. Therefore, we may consider the following question:

Question 7.4. Let (x_n) be an unconditional basis of X. To what extent can ideals generated by block bases of (x_n) determine the equivalence relation $E(X,(x_n))$?

It is worth noting that block bases of the unit bases of any space ℓ_p (in fact, any Orlicz sequence space) generate the same class of ideals, though they generate totally different equivalence relations with respect to Borel reducibility.

While (x_n) is a conditional basis of X, $\mathcal{I}(X,(x_n))$ is not an ideal in general. A powerful alternative tool is $\mathcal{S}(X,(x_n)) = \{\epsilon \in \{-1,1\}^{\mathbb{N}} : \sum_n \epsilon(n)x_n \text{ converges}\}$. In fact, it is already used in proofs of theorems 4.4, 5.11, and Lemma 5.8.

Besides unconditional bases, we are also interested in H.I. spaces (hereditarily indecomposable Banach spaces). Gowers [9] proved that any basis of a Banach space has a block basis which is either unconditional or a basis of an H.I. subspace. Then another interesting question is:

Question 7.5. Let (x_n) be an unconditional basis of X, (y_n) a basis of an H.I. space Y. Is it possible that $E(X,(x_n))$ and $E(Y,(y_n))$ are compatible in SER?

In contract, it is well known that, in this situation, no infinite dimensional Banach space can embed into both X and Y.

Acknowledgments. The main results of this paper were conducted while I visited Nanyang Technological University and Institute for Mathematical Sciences of the National University of Singapore. The article was written while I visited the University of North Texas. I would like to thank Guohua Wu, Chi Tat Chong, and Su Gao for invitations of these visits. I am grateful to Rui Liu, Xin Ma, Zhi Yin, Minggang Yu, and Yukun Zhang for conversations in seminars. Special thanks are due to Su Gao and Bunyamin Sari for useful suggestions.

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